Instructor's Solutions Manual

for

Introduction to Stochastic Processes with R

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Chapter 1

1.1 For the following scenarios identify a stochastic process $\{X_t, t \in I\}$, describing (i) X_t in context, (ii) state space, and (iii) index set. State whether the state space and index set are discrete or continuous.

b) X_t is the student's status at the end of year t. State space (discrete): $\mathcal{S} = \{\text{Drop Out, Frosh, Sophomore, Junior, Senior, Graduate}\}$. Index set (discrete): $I = \{0, 1, 2, \ldots\}$.

c) X_t is the magnitude for an earthquake which occurs at time t. State space (continuous): $(0, 10)$. Index set (continuous): $[0, \infty)$.

d) X_t is the circumference of the tree at location t. State space (continuous): $(0, \infty)$. Index set (continuous): $[0, 2] \times [0, 2]$; or the $x - y$ coordinates of a location in the arboretum.

e) X_t is the arrival time of student t. State space (continuous): $[0, 60]$. Index set (discrete): $\{1, 2, \ldots, 30\}$.

f) X_t is the order of the deck of cards after t shuffles. State space (discrete): Set of all orderings of the deck (52! elements). Index set (discrete): $\{0, 1, 2, \ldots\}$.

1.2 A regional insurance company insures homeowners against flood damage. Half of their policyholders are in Florida, 30% in Louisiana, and 20% in Texas.

a) Let A be the event that a claim is filed for flood damage. Then

$$\begin{aligned} P(A) &= P(A|F)P(F) + P(A|L)P(L) + P(A|T)P(T) \\ &= (0.03)(0.50) + (0.015)(0.30) + (0.02)(0.20) = 0.0235. \end{aligned}$$

b) $P(T|A) = P(A|T)P(T)/P(A) = (0.02)(0.20)/0.0235 = 0.17$.

1.3 Let $B_1, \ldots, B_k$ be a partition of the sample space. For events A and C, prove the *law of total probability for conditional probability.*

$$\begin{aligned} \sum_{i=1}^{k} P(A|B_i \cap C)P(B_i|C) &= \sum_{i=1}^{k} \left(\frac{P(A \cap B_i \cap C)}{P(B_i \cap C)} \right) \left(\frac{P(B_i \cap C)}{P(C)} \right) \\ &= \sum_{i=1}^{k} \frac{P(A \cap B_i \cap C)}{P(C)} = \frac{1}{P(C)} \sum_{i=1}^{k} P(A \cap B_i \cap C) \\ &= \frac{P(A \cap C)}{P(C)} = P(A|C). \end{aligned}$$

1.4 See Exercise 1.2. Among policyholders who live within 5 miles of the coast, 75% live in Florida, 20% live in Louisiana, and 5% live in Texas. Suppose a policyholder lives within 5 miles of the coast. Use the law of total probability for conditional probability to find the chance they will file a claim for flood damage next year.

Let A be the event that a claim is filed. Let C be the event that a claimant lives within five miles of the coast. Then

$$\begin{aligned} P(A|C) &= P(A|F,C)P(F|C) + P(A|L,C)P(L|C) + P(A|T,C)P(T|C) \\ &= (0.10)(0.75) + (0.06)(0.20) + (0.06)(0.05) = 0.09. \end{aligned}$$

1.5 Two fair, six-sided dice are rolled. Let X_1, X_2 be the outcomes of the first and second die, respectively.

a) Uniform on $\{1, 2, 3, 4, 5, 6\}$.

b) Uniform on $\{2, 3, 4, 5, 6\}$.

1.6 Bob has n coins in his pocket. One is two-headed, the rest are fair. A coin is picked at random, flipped, and shows heads.

Let A be the event that the coin is two-headed.

$$P(A|H) = \frac{P(H|A)P(A)}{P(H|A)P(A) + P(H|A^c)P(A^c)} = \frac{(1)(1/n)}{(1)(1/n) + (1/2)((n-1)/n)} = \frac{2}{n+1}.$$

1.7 A rat is trapped in a maze with three doors and some hidden cheese.

Let X denote the time until the rat finds the cheese. Let 1, 2, and 3 denote each door, respectively. Then

$$\begin{aligned} E(X) &= E(X|1)P(1) + E(X|2)P(2) + E(X|3)P(3) \\ &= (2 + E(X))\frac{1}{3} + (3 + E(X))\frac{1}{3} + (1)\frac{1}{3} = 2 + E(X)\frac{2}{3}. \end{aligned}$$

Thus, $E(X) = 6$ minutes.

1.8 A bag contains 1 red, 3 green, and 5 yellow balls. A sample of four balls is picked. Let G be the number of green balls in the sample. Let Y be the number of yellow balls in the sample.

a)

$$P(G = 1|Y = 2) = P(G = 2|Y = 2) = 1/2.$$

b) The conditional distribution of G is binomial with $n = 2$ and $p = 3/9 = 1/3$.

$$P(G = k|Y = 2) = \binom{2}{k}(1/3)^k(2/3)^{2-k}, \text{ for } k = 0, 1, 2.$$

1.9 Suppose X is uniformly distributed on $\{1, 2, 3, 4\}$. If $X = x$, then Y is uniformly distributed on $\{1, \ldots, x\}$.

a) $P(Y = 2|X = 2) = 1/2$.

b)

$$\begin{aligned} P(Y=2) &= P(Y=2|X=2)P(X=2) + P(Y=2|X=3)P(X=3) \\ &\quad + P(Y=2|X=4)P(X=4) = (1/2)(1/4) + (1/3)(1/4) + (1/4)(1/4) \\ &= 13/48. \end{aligned}$$

c) $P(X=2|Y=2) = P(X=2, Y=2)/P(Y=2) = (1/8)/(13/48) = 6/13$.

d) $P(X=2) = 1/4$.

e) $P(X=2, Y=2) = P(Y=2|X=2)P(X=2) = (1/2)(1/4) = 1/8$.

1.10 A die is rolled until a 3 occurs. By conditioning on the outcome of the first roll, find the probability that an even number of rolls is needed.

Let A be the event that an even number of rolls is needed. Let B be the event that a 3 occurs on the first roll.

$$P(A) = P(A|B)P(B) + P(A|B^c)P(B^c) = (0)(1/6) + (1 - P(A))(5/6)$$

gives $P(A) = 5/11$.

1.11 Consider gambler's ruin where at each wager, the gambler wins with probability p and loses with probability $q = 1-p$. The gambler stops when reaching \$$n$ or losing all their money. If the gambler starts with x, with $0 < x < n$, find the probability of eventual ruin.

Let x_k be the probability of reaching n when the gambler's fortune is k. Then

$$x_k = x_{k+1}p + x_{k-1}q, \text{ for } 1 \le k \le n-1,$$

with $x_0 = 0$ and $x_n = 1$, which gives

$$x_{k+1} - x_k = (x_k - x_{k-1})\frac{q}{p}, \text{ for } 1 \le k \le n-1.$$

It follows that

$$x_k - x_{k-1} = \cdots = (x_1 - x_0)(q/p)^{k-1} = x_1(q/p)^{k-1}, \text{ for all } k.$$

This gives $x_k - x_1 = \sum_{i=2}^{k} x_1(q/p)^{i-1}$. For $p \ne q$,

$$x_k = \sum_{i=1}^{k} x_1(q/p)^{i-1} = x_1 \frac{1-(q/p)^k}{1-q/p}.$$

For $k = n$, this gives

$$1 = x_n = x_1 \frac{1-(q/p)^n}{1-q/p}.$$

Thus $x_1 = (1-q/p)/(1-(q/p)^n)$, which gives

$$x_k = \frac{1-(q/p)^k}{1-(q/p)^n}, \text{ for } k = 0, \ldots, n.$$

For $p = q = 1/2$, $x_k = k/n$, for $k = 0, 1, \ldots, n$.

1.12 In n rolls of a fair die, let X be the number of ones obtained, and Y the number of twos. Find the conditional distribution of X given $Y = y$.

If $Y = y$, the number of 1s has a binomial distribution with parameters $n - y$ and $p = 1/5$.

1.13 Random variables X and Y have joint density

$$f(x, y) = 3y, \text{ for } 0 < x < y < 1.$$

a) The marginal density of X is

$$f_X(x) = \int_x^1 3y\, dy = 3(1 - x^2)/2, \text{ for } 0 < x < 1.$$

This gives

$$f_{Y|X}(y|x) = 2y/(1 - x^2), \text{ for } x < y < 1.$$

b) The marginal density of Y is

$$f_Y(y) = \int_0^y 3y\, dx = 3y^2, \text{ for } 0 < y < 1.$$

The conditional distribution of X given $Y = y$ is uniform on $(0, y)$.

1.14 Random variables X and Y have joint density function

$$f(x, y) = 4e^{-2x}, \quad \text{for } 0 < y < x < \infty.$$

a) The marginal density of Y is

$$f_Y(y) = \int_y^\infty 4e^{-2x}\, dx = 2e^{-2y}, \text{ for } y > 0.$$

This gives

$$f_{X|Y}(x|y) = 2e^{-2(x-y)}, \text{ for } y > x.$$

b) The conditional distribution of Y given $X = x$ is uniform on $(0, x)$.

1.15 Let X and Y be uniformly distributed on the disk of radius 1 centered at the origin.

The area of the circle is π. With $x^2 + y^2 = 1$, the joint density is

$$f(x, y) = \frac{1}{\pi}, \text{ for } -1 < x < 1, \ -\sqrt{1 - x^2} < y < \sqrt{1 - x^2}.$$

Integrating out the y term gives the marginal density

$$f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi}\, dy = \frac{2\sqrt{1 - x^2}}{\pi}, \text{ for } -1 < x < 1.$$

The conditional density is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{1/\pi}{2\sqrt{1-x^2}/\pi} = \frac{1}{2\sqrt{1-x^2}},$$

for $-1\sqrt{1-x^2} < y < \sqrt{1-x^2}$. The conditional density does not depend on y. The conditional distribution of Y given $X = x$ is uniform on $(-\sqrt{1-x^2}, \sqrt{1-x^2})$.

1.16 Find the expected number of aces in a poker hand given that the first card is an ace.

Let Y be the number of aces in a poker hand. Let A be the event that the first card is an ace. Consider $P(Y = y|A)$, for $y = 1, 2, 3, 4$. Given that the first card is an ace, there are $\binom{51}{4}$ ways to pick the remaining 4 cards. To obtain y aces in the hand, choose $y - 1$ aces from the remaining 3 aces, and $4 - (y - 1) = 5 - y$ non-aces from the 48 non-aces in the deck. This gives

$$E(Y|A) = \sum_{y=1}^{4} yP(Y = y|A) = \sum_{y=1}^{4} y \binom{3}{y-1}\binom{48}{5-y} \Big/ \binom{51}{4} = \frac{21}{17}.$$

1.17 Let X be a Poisson random variable with $\lambda = 3$. Find $E(X|X > 2)$.

$$E(X|X > 2) = \frac{1}{P(X > 2)} \sum_{k=3}^{\infty} k \frac{e^{-3}3^k}{k!} = 4.16525.$$

1.18 From the definition of conditional expectation given an event, show $E(I_B|A) = P(B|A)$.

$$E(I_B|A) = \frac{1}{P(A)} E(I_B I_A) = \frac{1}{P(A)} E(I_{B\cap A}) = \frac{1}{P(A)} P(B \cap A) = P(B|A).$$

1.19 As in Example 1.21,

$$\begin{aligned} E(Y^2) &= E(Y^2|T)P(T) + E(Y^2|HT)P(HT) + E(Y^2|TT)P(TT) \\ &= E((1+Y)^2)\frac{1}{2} + E((2+Y)^2)\frac{1}{4} + (4)\frac{1}{4} \\ &= (1 + 2E(Y) + E(Y^2))\frac{1}{2} + (4 + 4E(Y) + E(Y^2))\frac{1}{4} + 1 \\ &= \frac{5}{2} + 12 + E(Y^2)\frac{3}{4}. \end{aligned}$$

This gives $E(Y^2) = 58$, and thus,

$$Var(Y) + E(Y^2) - E(Y)^2 = 58 - 36 = 22.$$

1.20 A fair coin is flipped repeatedly.

a) Let Y be the number of flips needed. Then

$$\begin{aligned} E(Y) &= E(Y|T)(1/2) + E(Y|HT)(1/4) + E(Y|HHT)(1/8) + E(Y|HHH)(1/8) \\ &= (1 + E(Y))(1/2) + (2 + E(Y))(1/4) + (3 + E(Y))(1/8) + 3(1/8) \\ &= E(Y)(7/8) + 7/4. \end{aligned}$$

Solving for $E(Y)$ gives $E(Y) = 14$.

b) Generalizing a) gives

$$E(Y) = \sum_{i=1}^{k} (i + E(Y))(1/2^i) + k(1/2^k) = 2 - 2^{1-k} + E(Y)(1 - 2^{-k}).$$

Solving for $E(Y)$ gives

$$E(Y) = 2^{k+1} - 2.$$

1.21 Let T be a nonnegative, continuous random variable. Show $E(T) = \int_0^\infty P(T > t)\, dt$.

$$\begin{aligned} \int_0^\infty P(T > t)\, dt &= \int_0^\infty \int_t^\infty f(s)\, ds\, dt = \int_0^\infty \int_0^s f(s)\, dt\, ds \\ &= \int_0^\infty s f(s)\, ds = E(T). \end{aligned}$$

1.22 Find $E(Y|X)$ when (X, Y) is uniformly distributed on several regions.

a) The conditional distribution of Y given $X = x$ is uniformly distributed on $[c, d]$. Thus $E(Y|X = x) = (c + d)/2$, for all x. This gives $E(Y|X) = (c + d)/2$.

b) The area of the triangle is $1/2$. Thus the joint density of X and Y is

$$f(x, y) = 2, \text{ for } 0 < y < x < 1.$$

Integrating out the y term gives

$$f_X(x) = 2x, \text{ for } 0 < x < 1.$$

The conditional density of Y given $X = x$ is

$$f_{Y|X}(y|x) = \frac{2}{2x} = \frac{1}{x}, \text{ for } 0 < y < x.$$

The conditional distribution of Y given $X = x$ is uniform on $(0, x)$. Hence $E(Y|X = x) = x/2$, giving $E(Y|X) = X/2$.

c) $E(Y|X) = 0$.

1.23 Let $X_1, X_2, \ldots$ be an i.i.d. sequence of random variables with common mean μ. Let $S_n = X_1 + \cdots + X_n$, for $n \geq 1$.

a) For $m \leq n$,

$$E(S_m|S_n) = E(X_1 + \cdots + X_m|S_n) = \sum_{i=1}^{m} E(X_i|S_n) = mE(X_1|S_n).$$

Also,

$$S_n = E(S_n|S_n) = E(X_1 + \cdots + X_n|S_n) = \sum_{i=1}^{n} E(X_1|S_n) = nE(X_1|S_n).$$

This gives $E(S_m|S_n) = mE(X_1|S_n) = mS_n/n$.

b) For $m > n$,

$$\begin{aligned} E(S_m|S_n) &= E(S_n + X_{n+1} + \cdots + X_m|S_n) \\ &= E(S_n|S_n) + E(X_{n+1} + \cdots + X_m|S_n) \\ &= S_n + \sum_{i=n+1}^{m} E(X_i|S_n) = S_n + \sum_{i=n+1}^{m} E(X_i) = S_n + (m-n)\mu. \end{aligned}$$

1.24 Prove the law of total expectation $E(Y) = E(E(Y|X))$ for the continuous case.

$$\begin{aligned} E(E(Y|X)) &= \int_{-\infty}^{\infty} E(Y|X=x) f_X(x)\,dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y f_{Y|X}(y|x)\,dy f_X(x)\,dx \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y \frac{f(x,y)}{f_X(x)} f_X(x)\,dy\,dx = \int_{-\infty}^{\infty} y \int_{-\infty}^{\infty} f(x,y)\,dx\,dy \\ &= \int_{-\infty}^{\infty} y f_Y(y)\,dy = E(Y). \end{aligned}$$

1.25 Let X and Y be independent exponential random variables with respective parameters 1 and 2. Find $P(X/Y < 3)$ by conditioning.

$$\begin{aligned} P(X/Y < 3) &= \int_0^{\infty} P(X/Y < 3|Y = y) 2e^{-2y}\,dy = \int_0^{\infty} P(X < 3y) 2e^{-2y}\,dy \\ &= \int_0^{\infty} \left(1 - e^{-3y}\right) 2e^{-2y}\,dy = \frac{3}{5}. \end{aligned}$$

1.26 The density of X is $f(x) = xe^{-x}$, for $x > 0$. Given $X = x$, Y is uniformly distributed on $(0, x)$. Find $P(Y < 2)$ by conditioning on X.

$$
\begin{aligned}
P(Y < 2) &= \int_0^\infty P(Y < 2|X = x)xe^{-x}\,dx = \int_0^2 (1)xe^{-x}\,dx + \int_2^\infty \frac{2}{x}xe^{-x}\,dx \\
&= (1 - 3e^{-2}) + 2e^{-2} = 1 - e^{-2}.
\end{aligned}
$$

1.27 A restaurant receives N customers per day, where N is a random variable with mean 200 and standard deviation 40. The amount spent by each customer is normally distributed with mean \$15 and standard deviation \$3. Find the mean and standard deviation of the total spent per day.

Let T be the total spent at the restaurant. Then $E(T) = 200(15) = \$3000$. and

$$
Var(T) = 9(200) + 15^2(40^2) = 361800, \text{ and } SD(T) = \$601.50.
$$

1.28 The number of daily accidents has a Poisson distribution with parameter Λ, where Λ is itself a random variable. Find the mean and variance of the number of accidents per day when Λ is uniformly distributed on $(0, 3)$.

Let X be the number of accidents per day. The conditional distribution of X given $\Lambda = \lambda$ is Poisson with parameter λ. Thus, $E(X|\Lambda) = \Lambda$ and $Var(X|\Lambda) = \Lambda$. This gives

$$
E(X) = E(E(X|\Lambda)) = E(\Lambda) = \frac{3}{2}
$$

and

$$
Var(X) = E(Var(X|\Lambda)) + Var(E(X|\Lambda)) = E(\Lambda) + Var(\Lambda) = \frac{3}{2} + \frac{9}{12} = \frac{9}{4}.
$$

1.29 If X and Y are independent, does $Var(Y|X) = Var(Y)$?

Yes, as $Var(Y|X) = E(Y^2|X) - E(Y|X)^2 = E(Y^2) - E(Y)^2 = Var(Y)$.

1.30 Suppose $Y = g(X)$ is a function of X.

a) $E(Y|X = x) = E(g(x)|X = x) = g(x)$, for all x. Thus $E(Y|X) = Y$.

b) $Var(Y|X = x) = Var(g(x)|X = x) = 0$, for all x. Thus $Var(Y|X) = 0$.

1.31 Consider a sequence of i.i.d. Bernoulli trials with success parameter p. Let X be the number of trials needed until the first success occurs. Then X has a geometric distribution with parameter p. Find the variance of X by conditioning on the first trial.

By conditioning on the first trial,

$$E(X^2) = (1)p + E\left((1+X)^2\right)(1-p) = p + E(1+2X+X^2)(1-p)$$
$$= 1 + \frac{2(1-p)}{p} + E(X^2)(1-p) = \frac{2-p}{p^2} + E(X^2)(1-p),$$

which gives $E(X^2) = (2-p)/p^2$, and

$$Var(X) = E(X^2) - E(X)^2 = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

1.32 `R`: Simulate flipping three fair coins and counting the number of heads X.

a) Use your simulation to estimate $P(X = 1)$ and $E(X)$.

```
> trials <- 100000
> sim1 <- numeric(trials)
> sim2 <- numeric(trials)
> for (i in 1:trials) {
+ coins <- sample(0:1,3,replace=T)
+ sim1[i] <- if (sum(coins)==1) 1 else 0
+ sim2[i] <- sum(coins)
+ }
> mean(sim1) # P(X=1)
[1] 0.37673
> mean(sim2) # E(X)
[1] 1.50035
```

b) Modify the above to allow for a biased coin where $P(\text{Heads}) = 3/4$.

```
> trials <- 100000
> sim1 <- numeric(trials)
> sim2 <- numeric(trials)
> for (i in 1:trials) {
+ coins <- sample(c(0,1),3,prob=c(1/4,3/4),replace=T)
+ sim1[i] <- if (sum(coins)==1) 1 else 0
+ sim2[i] <- sum(coins)
+ }
> mean(sim1) # P(X=1)
[1] 0.14193
> mean(sim2) # E(X)
[1] 2.24891
```

1.33 `R`: Cards are drawn from a standard deck, with replacement, until an ace appears. Simulate the mean and variance of the number of cards required.

```
> trials <- 10000
> # let 1,2,3,4 denote the aces
```

```
> simlist <- numeric(trials)
> for (i in 1:trials) {
+  ct <- 0
+ ace <- 0
+ while (ace==0) {
+  card <- sample(1:52,1,replace=T)
+  ct <- ct + 1
+  if (card <= 4) ace <- 1
+  }
+  simlist[i] <- ct
+  }
> mean(simlist)
[1] 13.1465
> var(simlist)
[1] 154.3741
```

1.34 `R`: The time until a bus arrives has an exponential distribution with mean 30 minutes.

a) Use the command `rexp()` to simulate the probability that the bus arrives in the first 20 minutes.

```
> trials <- 1000000
> sum( rexp(trials,1/30) < 20)/trials
[1] 0.486829
```

b) Use the command `pexp()` to compare with the exact probability.

```
> pexp(20,1/30)
[1] 0.4865829
```

1.35 `R`: See the script file **gamblersruin.R**. A gambler starts with a $60 initial stake.

a) The gambler wins, and loses, each round, with probability 0.5. Simulate the probability the gambler wins $100 before he loses everything.

```
> trials <- 1000
> simlist <- replicate(trials,gamble(60,100,0.5))
> 1-mean(simlist)
[1] 0.609
```

b) The gambler wins each round with probability 0.51. Simulate the probability the gambler wins $100 before he loses everything.

```
> simlist <- replicate(trials,gamble(60,100,0.51))
> 1-mean(simlist)
[1] 0.928
```

1.36 `R`: See the script file **ReedFrost.R**. Observe the effect on the course of the disease by changing the initial values for the number of people susceptible and infected.

1.37 `R`: Highway accidents. Simulate the results of Exercise 1.28. Estimate the mean and variance of the number of accidents per day.

```
> trials <- 100000
> sim <- replicate(trials,rpois(1,runif(1,0,3)))
> mean(sim)
[1] 1.50068
> var(sim)
[1] 2.256382
```

Chapter 2

2.1 Find the following:

a) $P(X_7 = 3|X_6 = 2) = P_{2,3} = 0.6$.

b) $P(X_9 = 2|X_1 = 2, X_5 = 1, X_7 = 3) = P(X_9 = 2|X_7 = 3) = P^2_{3,2} = 0.27$;

c) $P(X_0 = 3|X_1 = 1) = P(X_1 = 1|X_0 = 3)P(X_0 = 3)/P(X_1 = 1) = P_{31}\alpha_3/(\alpha P)_1 = (0.3)(0.5)/(0.17) = 15/17 = 0.882$;

d) $E(X_2) = \sum_{k=1}^{3} kP(X_2 = k) = (0.182, 0.273, 0.545) \cdot (1, 2.3) = 2.363$.

2.2 a) $P(X_2 = 1|X_1 = 3) = P_{3,1} = 1/3$.

b) $P(X_1 = 3, X_2 = 1) = P(X_2 = 1|X_1 = 3)P(X_1 = 3) = P_{3,1}(\alpha P)_3 = (1/3)(5/12) = 5/36$.

c) $P(X_1 = 3|X_2 = 1) = P(X_1 = 3, X_2 = 1)/P(X_2 = 1) = (5/36)/(\alpha P^2)_1 = (5/36)/(5/9) = 1/4$.

d) $P(X_9 = 1|X_1 = 3, X_4 = 1, X_7 = 2) = P(X_9 = 1|X_7 = 2) = P^2_{2,1} = 0$.

2.3 Consider the Wright-Fisher model with $k = 3$ genes. If the population initially has one A allele, find the probability that there are no A alleles in three generations.

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 8/27 & 4/9 & 2/9 & 1/27 \\ 1/27 & 2/9 & 4/9 & 8/27 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array} \text{ and } \boldsymbol{P}^3 = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0.517 & 0.154 & 0.143 & 0.187 \\ 0.187 & 0.143 & 0.154 & 0.517 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

The desired probability is $P^3_{10} = 0.517$.

2.4 For the general two-state chain,

a)

$$\boldsymbol{P}^2 = \begin{pmatrix} 1 + p(p + q - 2) & p(2 - p - q) \\ q(2 - p - q) & 1 + q(p + q - 2) \end{pmatrix}.$$

b) Distribution of X_1 is $(\alpha_1(1 - p) + \alpha_2 q, \alpha_2(1 - q) + \alpha_1 p)$.

2.5 For *random walk with reflecting boundaries,* suppose $k = 3$, $q = 1/4$, $p = 3/4$, and the initial distribution is uniform.

a)

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/4 & 0 & 3/4 & 0 \\ 0 & 1/4 & 0 & 3/4 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

b) $P(X_7 = 1|X_0 = 3, X_2 = 2, X_4 = 2) = P^3_{21} = 19/64 = 0.297$.

c) $P(X_3 = 1, X_5 = 3) = (\alpha P^3)_1 P^2_{13} = \left(\frac{47}{256}\right)\left(\frac{9}{16}\right) = 0.103$.

2.6 A tetrahedron die has four faces labeled 1, 2, 3, and 4. In repeated independent rolls of the die, let $X_n = \max\{X_0, \ldots, X_n\}$ be the maximum value after n rolls.

a)

$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 0 & 1/2 & 1/4 & 1/4 \\ 0 & 0 & 3/4 & 1/4 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

b) With $\alpha = (1/4, 1/4, 1/4, 1/4)$,

$$P(X_3 \geq 3) = (\alpha P^3)_3 + (\alpha P^3)_4 = (65/256) + (175/256) = 15/16.$$

2.7

$$\begin{aligned} &P(Y_{n+1} = j | Y_n = i, Y_{n-1} = i_{n-1}, \ldots, Y_0 = i_0) \\ &\qquad = P(X_{3n+3} = j | X_{3n} = i, X_{3n-3} = i_{n-1}, \ldots, X_0 = i_0) \\ &\qquad = P(X_{3n+3} = j | X_{3n} = i) = P(X_3 = j | X_0 = i) \\ &\qquad = P(Y_1 = j | Y_0 = i) = P^3_{ij}. \end{aligned}$$

Thus, (Y_n) is a time-homogenous Markov chain with transition matrix is $\boldsymbol{P}^3$.

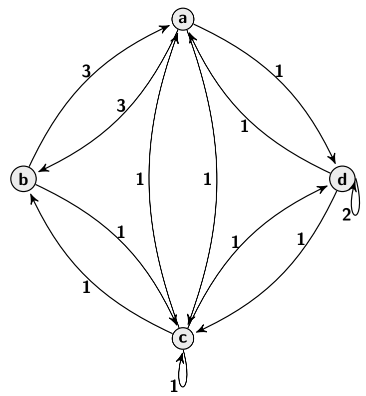
2.8 Give the Markov transition matrix for random walk on the weighted graph.

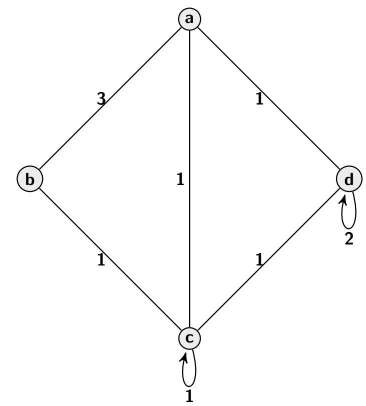
$$\boldsymbol{P} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 1/6 & 1/2 & 0 & 1/3 \\ 1/10 & 1/5 & 1/5 & 1/10 & 2/5 \\ 1/2 & 1/3 & 0 & 1/6 & 0 \\ 0 & 1/2 & 1/2 & 0 & 0 \\ 1/3 & 2/3 & 0 & 0 & 0 \end{pmatrix} \end{array}.$$

2.9 Give the transition matrix for the transition graph.

$$\boldsymbol{P} = \begin{array}{c} \\ a \\ b \\ c \\ d \\ e \end{array} \begin{array}{c} \begin{array}{ccccc} a & b & c & d & e \end{array} \\ \begin{pmatrix} 0 & 0 & 3/5 & 0 & 2/5 \\ 1/7 & 2/7 & 0 & 0 & 4/7 \\ 0 & 2/9 & 2/3 & 1/9 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 3/4 & 0 & 0 & 1/4 & 0 \end{pmatrix} \end{array}.$$

2.10 Exhibit the directed, and undirected, weighted transition graphs.





2.11 You start with five dice. Roll all the dice and put aside those dice which come up 6. Then roll the remaining dice, putting aside those dice which come up 6. And so on. Let X_n be the number of dice which are sixes after n rolls.

a)

$$P_{ij} = \binom{5-i}{j-i} \left(\frac{1}{6}\right)^{j-i} \left(\frac{5}{6}\right)^{5-j}, \text{ for } 0 \leq i, j \leq 5.$$

b) $P^3_{0,5} = 0.01327$.

c) $\boldsymbol{P}^{100}$ is a matrix (approximately) of all 0s with 1s in the last column.

2.12 Bernoulli-Laplace model. The number of blue balls in the left urn at time n depends only on the number of blue balls in the left urn at time $n-1$ and the current balls which are selected.

If there are i blue balls in the left urn, there are $k-i$ red balls in that urn. Also there are $k-i$ blue balls and i red balls in the right urn. This gives

$$P_{ij} = \begin{cases} i^2/k^2, & \text{if } j = i-1 \\ 2i(k-i)/k^2, & \text{if } j = i \\ (k-i)^2/k^2, & \text{if } j = i+1. \end{cases}$$

2.13 Transposition scheme transition matrix for 3 books is

$$\boldsymbol{P} = \begin{array}{c} \\ abc \\ acb \\ bac \\ bca \\ cab \\ cba \end{array} \begin{array}{c} \begin{array}{cccccc} abc & acb & bac & bca & cab & cba \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 & 0 & 0 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 & 0 \\ 1/3 & 0 & 1/3 & 1/3 & 0 & 0 \\ 0 & 0 & 1/3 & 1/3 & 0 & 1/3 \\ 0 & 1/3 & 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 0 & 1/3 & 1/3 & 1/3 \end{pmatrix} \end{array}.$$

2.14 There are k songs on Mary's music player. The player is set to *shuffle* mode, which plays songs uniformly at random, sampling with replacement. Let X_n denote the number of *unique* songs which have been heard after the nth play.

a) The transition matrix is

$$P_{ij} = \begin{cases} i/n, & \text{if } j = i \\ (n-i)/n, & \text{if } j = i+1. \end{cases}$$

b) With $n = 4$, the desired probability is $P^6_{0,4} = 195/512 = 0.381$.

2.15 Let $Z_n = (X_{n-1}, X_n)$. The sequence $Z_0, Z_1, \ldots$ is a Markov chain with state space $\mathcal{S} \times \mathcal{S} = \{(0,0), (0,1), (1,0), (1,1)\}$. Transition matrix is

$$\boldsymbol{P} = \begin{matrix} & (0,0) & (0,1) & (1,0) & (1,1) \\ (0,0) & 1-p & p & 0 & 0 \\ (0,1) & 0 & 0 & q & 1-q \\ (1,0) & 1-p & p & 0 & 0 \\ (1,1) & 0 & 0 & q & 1-q \end{matrix}.$$

2.16 Suppose $\boldsymbol{P}$ is a stochastic matrix with equal rows, with $P_{ij} = p_j$. By induction on n,

$$P^n_{ij} = \sum_k P^{n-1}_{ik} P_{kj} = \sum_k p_k p_j = p_j.$$

2.17 Let $\boldsymbol{P}$ be a stochastic matrix. Show that $\lambda = 1$ is an eigenvalue of $\boldsymbol{P}$.

Let $\mathbf{1}$ be a column vector of all 1s. Then the jth component of $\boldsymbol{P}\mathbf{1}$ is the sum of the jth row of $\boldsymbol{P}$. This gives $\boldsymbol{P}\mathbf{1} = \mathbf{1}$. Thus $\lambda = 1$ is an eigenvalue corresponding to the eigenvector $\mathbf{1}$.

2.18 Let $X_0, X_1, \ldots$ be a Markov chain on $\{1, \ldots, k\}$ with doubly stochastic transition matrix and uniform initial distribution. Show that the distribution of X_n is uniform.

$$(\alpha P)_j = \sum_i \alpha_i P_{ij} = \sum_i \frac{1}{k} P_{ij} = \frac{1}{k} \sum_i P_{ij} = \frac{1}{k}, \text{ for all } j.$$

2.19 Let

$$\boldsymbol{Q} = (1-p)\boldsymbol{I} + p\boldsymbol{P}, \text{ for } 0 < p < 1.$$

The entries of Q are nonnegative, and

$$\sum_j Q_{ij} = (1-p) \sum_j I_{ij} + p \sum_j P_{ij} = (1-p)(1) + p(1) = 1.$$

Consider a biased coin whose heads probability is p. When the chain is at state i, flip the coin. If tails, stay put. If heads, move according to the $\boldsymbol{P}$-chain.

Show that $\boldsymbol{Q}$ is a stochastic matrix. Give a probabilistic interpretation for the dynamics of a Markov chain governed by the $\boldsymbol{Q}$ matrix in terms of the original Markov chain. Use the imagery of flipping coins.

2.20 Let $Y_n = g(X_n)$, for $n \geq 0$. Show that $Y_0, Y_1, \ldots$ is not a Markov chain.

Consider

$$\begin{aligned} P(Y_2 = 0|Y_0 = 0, Y_1 = 1) &= P(g(X_2) = 0|g(X_0) = 0, g(X_1) = 1) \\ &= P(X_2 = 1|X_0 = 1, X_1 \in \{2, 3\}) \\ &= P(X_2 = 1|X_0 = 1, X_1 = 2) = 0. \end{aligned}$$

On the other hand,

$$\begin{aligned} P(Y_2 = 0|Y_0 = 1, Y_1 = 1) &= P(g(X_2) = 0|g(X_0) = 1, g(X_1) = 1) \\ &= P(X_2 = 1|X_0 \in \{2, 3\}, X_1 \in \{2, 3\}) \\ &= P(X_2 = 1|X_0 = 2, X_1 = 3) = P(X_2 = 1|X_1 = 3) = p. \end{aligned}$$

2.21 Let P and Q be two transition matrices on the same state space. We will define two processes, both started in some initial state i.

Process #1 is not a Markov chain as the distribution after n steps depends on the initial flip of the coin. Process #2 is a Markov chian. The i, j-th entry of its transition matrix is $(1/2)P_{ij} + (1/2)Q_{ij}$.

2.22 a) Base case ($n = 1$): $1 = 1^2$. By the induction hypothesis,

$$1 + 3 + \cdots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2,$$

which establishes the result.

b) Base case ($n = 1$): $1^2 = 1(2)(3)/6 = 1$. By the induction hypothesis,

$$\begin{aligned} \sum_{k=1}^{n+1} k^2 &= \left(\sum_{k=1}^{n} k^2\right) + (n + 1)^2 \\ &= \frac{n(n + 1)(2n + 1)}{6} + \frac{6n^2 + 12n + 6}{6} \\ &= \frac{(n + 1)(n + 2)2(n + 1) + 1)}{6}, \end{aligned}$$

which establishes the result.

c) Base case ($n = 1$): $(1 + x)^1 \geq 1 + x$. By the induction hypothesis, using the fact that since $x > -1$, $1 + x > 0$,

$$\begin{aligned} (1 + x)^{n+1} = (1 + x)^n(1 + x) &\geq (1 + nx)(1 + x) \\ &= 1 + nx + x + nx^2 = 1 + (n + 1)x + nx^2 \geq 1 + (n + 1)x, \end{aligned}$$

which establishes the result.

2.23 `R`: Simulate the first 20 letters of the Pushkin poem Markov chain.

```
> mat <- matrix(c(0.175,0.526,0.825,0.474),nrow=2)
> markov(c(0.5,0.5),mat,20,c("v","c"))
 [1] "c" "c" "c" "c" "v" "c" "v" "c" "c" "c" "c" "v" "c" "c"
[15] "v" "v" "v" "c" "c" "c" "v"
```

2.24 `R`: Simulate 50 steps of the random walk on the graph in Figure 2.1.

```
> mat <- matrix(c(0,1/4,0,0,0,0,1,0,1/4,1/4,1/3,0,0,1/4,0,1/4,1/3,1/2,
+   0,1/4,1/4,0,1/3,1/2,0,1/4,1/4,1/4,0,0,0,0,1/4,1/4,0,0),nrow=6)
> markov(rep(1/6,6),mat,50,c("a","b","c","d","e","f"))
 [1] "a" "b" "e" "c" "e" "b" "c" "f" "c" "b" "c" "d" "f" "d"
[15] "c" "e" "d" "b" "e" "b" "e" "d" "c" "d" "c" "d" "e" "b"
[29] "c" "e" "b" "c" "d" "c" "b" "e" "c" "b" "e" "c" "e" "c"
[43] "b" "e" "b" "e" "d" "b" "a" "b" "d"
```

Long-term probability of visiting vertex c is $\pi_c = 4/18$.

2.25 `R`: Use technology to estimate the long-term distribution of dolphin activity.

Socializing	Traveling	Milling	Feeding	Resting
0.148	0.415	0.096	0.216	0.125

2.26 `R`:

a) Least likely: Port 80. $P(X_2 = 80|X_0 = \text{No}) = 0.031$.

Most likely: Port 139. $P(X_2 = 139|X_0 = \text{No}) = 0.387$.

b)

80	135	139	445	No
0.021	0.267	0.343	0.227	0.141

2.27 `R`: Simulate gambler's ruin for a gambler with initial stake \$2, playing a fair game. Estimate the probability that the gambler is ruined before he wins \$5.

```
> # a) Uses gamble function in gamblersruin.R
> trials <- 10000
> simlist <- replicate(trials,gamble(2,5,0.5))
> mean(simlist)
[1] 0.6055
>
> # b,c) Uses matrixpower function in utilities.R
> mat <- matrix(c(1,1/2,0,0,0,0,0,0,1/2,0,0,0,0,1/2,0,1/2,0,0,0,0,
+   1/2,0,1/2,0,0,0,0,1/2,0,0,0,0,0,0,1/2,1),nrow=6)
> rownames(mat) <- 0:5
> colnames(mat) <- 0:5
> round(matrixpower(mat,100),5)
    0 1 2 3 4   5
0 1.0 0 0 0 0 0.0
1 0.8 0 0 0 0 0.2
2 0.6 0 0 0 0 0.4
3 0.4 0 0 0 0 0.6
4 0.2 0 0 0 0 0.8
5 0.0 0 0 0 0 1.0
> # desired probability is (n-k)/n = (5-2)/5
```

Chapter 3

3.1 Stationary distribution is

$$\boldsymbol{\pi} = \left(\frac{1}{5}, \frac{1}{3}, \frac{2}{5}, \frac{1}{15}\right).$$

3.2 Show doubly stochastic matrix has uniform stationary distribution.

Let $\boldsymbol{\pi} = (1/k, \ldots, 1/k)$. For all j,

$$\sum_{i=1}^{k} \pi_i P_{ij} = \frac{1}{k}\sum_{i=1}^{k} P_{ij} = \frac{1}{k}.$$

3.3 $\boldsymbol{P}$ and $\boldsymbol{R}$ are regular. $\boldsymbol{Q}$ is regular for $0 < p < 1$.

3.4 Stationary distribution is

$$\boldsymbol{\pi} = \frac{1}{ab + ac + bc}(bc, ac, ab).$$

3.5 a) All nonnegative vectors of the form (a, a, b, c, a), where $3a+b+c = 1$, are stationary distributions.

b)

$$\lim_{n\to\infty} \boldsymbol{P}^n = \begin{pmatrix} 1/3 & 1/3 & 0 & 0 & 1/3 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 0 & 1/3 \end{pmatrix}.$$

If the chain starts in 1, 2, or 5, it eventually has equal chance of hitting 1, 2, or 5. If the chain starts in either 3 or 4, it stays in that state forever.

c) The long-term behavior of the chain depends on the initial state, therefore it does not have a limiting distribution. This does not contradict the fact that a limiting matrix exists, as that limiting matrix does not have equal rows.

3.6 a) The vector $\boldsymbol{x}$ defined by $x_k = 1/k!$, for $k \geq 1$, satisfies $\boldsymbol{xP} = \boldsymbol{x}$. The stationary distribution is

$$\pi_k = \frac{1}{k!(e-1)}, \text{ for } k = 1, 2, \ldots.$$

b) The chain is irreducible. All states are recurrent.

c) A stationary distribution does not exist.

3.7 A Markov chain has n states. If the chain is at state k, a biased coin is flipped, whose heads probability is p. If the coin lands heads, the chain stays at k. If the coin lands tails, the chain moves to a different state uniformly at random. The transition matrix has p's on the main diagonal. All other entries are $(1 - p)/(k - 1)$. The matrix is doubly stochastic and the stationary distribution is uniform.

3.8 a) The chain does not have a unique stationary distribution. All probability vectors of the form $(2a/3, a, b, b)$ are stationary distributions.

b)

$$\lim_{n\to\infty} \boldsymbol{P}^n = \begin{pmatrix} 3/5 & 2/5 & 0 & 0 \\ 3/5 & 2/5 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1/2 & 1/2 \end{pmatrix}.$$

c) The chain does not have a limiting distribution.

3.9 a) Yes.

b) No. Counter-example: $\boldsymbol{P} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

3.10 a) $\lim_{n\to\infty} P(X_n = j|X_{n-1} = i) = \lim_{n\to\infty} P(X_1 = j|X_0 = i) = P_{ij}$

b) $\lim_{n\to\infty} P(X_n = j|X_0 = i) = \pi_j$

c) $\lim_{n\to\infty} P(X_{n+1} = k, X_n = j|X_0 = i) = \lim_{n\to\infty} P(X_{n+1} = k|X_n = j)P(X_n = j|X_0 = i) = P_{jk}\pi_j$.

d) $\lim_{n\to\infty} P(X_0 = j|X_n = i) = \lim_{n\to\infty} P(X_n = i|X_0 = j)P(X_0 = j)/P(X_n = i) = \pi_i\pi_j/\pi_i = \pi_j$.

3.11 Consider a simple symmetric random walk on $\{0, 1, \ldots, k\}$ with reflecting boundaries.

a) Stationary distribution is

$$\pi_j = \begin{cases} 1/(2k), & \text{if } j = 0, k \\ 1/k, & \text{if } j = 1, \ldots, k-1. \end{cases}$$

b) $1/\pi_0 = 2k = 2000$ steps.

3.12 The set of all stationary distributions is the set of all probability vectors of the form

$$\left(\frac{s}{3s+2t}, \frac{t}{3s+2t}, \frac{2s}{3s+2t}, \frac{t}{3s+2t}\right), \text{ for } s, t \geq 0.$$

3.13 Find the communication classes. Rewrite the transition matrix in canonical form.

Communication classes are $\{4\}$ (recurrent, absorbing); $\{1, 5\}$ (recurrent); $\{2, 3\}$ (transient). All states have period 1.

$$\boldsymbol{P} = \begin{array}{c} \\ 2 \\ 3 \\ 1 \\ 5 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 2 & 3 & 1 & 5 & 4 \end{array} \\ \begin{pmatrix} 1/2 & 1/6 & 1/3 & 0 & 0 \\ 1/4 & 0 & 0 & 1/4 & 1/2 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

3.14 a) Long-term probability that a given day will be an episode day is 0.489.

b) Over a year's time the expected number of episode days is $(0.489362)(365) = 178.62$.

c) In the long-term, the average number of days that will transpire between episode days is $1/0.489 = 2.04$.

3.15 Simple random walk for random knight. Construct a graph on 64 vertices corresponding to a chessboard with an edge joining two "squares" if it is an allowable knight move. One finds the following degree distribution

Degree	Number of squares
2	4
3	8
4	20
6	16
8	16

The sum of the vertex degrees is $2(4) + 3(8) + 4(20) + 6(16) + 8(16) = 336$. Thus the expected number of steps to return from a corner square (of degree 2) is $336/2 = 168$.

3.16 By a similar derviation as with the knight one finds the following expected return times. Bishop: 40; Rook: 64; Queen: $1456/21 = 69.33$; King: 140.

3.17 Obtain a closed form expression for $\boldsymbol{P}^n$. Exhibit the matrix $\sum_{n=0}^{\infty} \boldsymbol{P}^n$ (some entries may be $+\infty$).

$$\boldsymbol{P}^n = \begin{pmatrix} 1/2^n & 1 - 1/2^n \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \sum_{n=0}^{\infty} \boldsymbol{P}^n = \begin{pmatrix} 2 & +\infty \\ 0 & +\infty \end{pmatrix}.$$

State 1 is transient. State 2 is recurrent.

3.18 Use first-step analysis to find the expected return time to state b.

Let $f_x = E[T_b | X_0 = x]$, for $x = a, b, c$. We have

$$\begin{aligned} f_b &= (1/4)(1 + f_a) + (3/4)(1 + f_c) \\ f_a &= (1/2)(1 + f_a) + (1/2) \\ f_c &= (1/2)(1 + f_a) + (1/2). \end{aligned}$$

This gives $f_a = f_c = 2$ and $f_b = 3$.

3.19 Use first-step analysis to find the expected time to hit d for the walk started in a.

Let p_x be the expected time to hit d for the walk started in x. By symmetry, $p_b = p_e$ and $p_c = p_e$. Solve

$$\begin{aligned} p_a &= \frac{1}{2}(1 + p_b) + \frac{1}{2}(1 + p_c) \\ p_b &= \frac{1}{2}(1 + p_a) + \frac{1}{2}(1 + p_c) \\ p_c &= \frac{1}{4}(1 + p_b) + \frac{1}{4} + \frac{1}{4}(1 + p_a) + \frac{1}{4}(1 + p_c) \end{aligned}$$

This gives $p_a = 10$.

3.20 Show that simple symmetric random walk on $\mathbb{Z}^2$, that is, on the integer points in the plane, is recurrent.

The walk moves in one of four directions with equal probability. As in the 1-dimensional case, consider P_{00}^{2n}, where 0 denotes the origin. Paths of length $2n$ return to the origin if and only if the path moves k steps to the left, k steps to the right, $n-k$ steps up, and $n-k$ steps down, for some $0 \leq k \leq n$. This gives

$$P_{00}^{2n} = \sum_{k=0}^{n} \frac{(2n)!}{k!k!(n-k)!(n-k)!} \frac{1}{4^{2n}} = \frac{1}{4^{2n}} \binom{2n}{n} \sum_{k=0}^{n} \binom{n}{k}^2 = \frac{1}{4^{2n}} \binom{2n}{n}^2.$$

In the one-dimensional case, it is shown by Stirling's approximation that $\binom{2n}{n}1/4^n \sim 1/\sqrt{n\pi}$, as $n \to \infty$. It follows that $\frac{1}{4^{2n}}\binom{2n}{n}^2 \sim 1/(n\pi)$ and $\sum_{n=1}^{\infty} 1/(n\pi) = \infty$, which gives recurrence.

3.21 Show that simple symmetric random walk on $\mathbb{Z}^3$ is transient.

The walk moves in one of six directions with equal probability. Paths of length $2n$ return to the origin if and only if the path moves j steps to the left, j steps to the right, k steps up, k steps down, $n-j-k$ steps forward, and $n-j-k$ steps backward, for some $n0 \leq j+k \leq n$. This gives

$$\begin{aligned} P_{00}^{2n} &= \frac{1}{6^{2n}} \sum_{0 \leq j+k \leq n} \frac{(2n)!}{j!j!k!k!(n-j-k)!(n-j-k)!} \\ &= \frac{1}{2^{2n}} \binom{2n}{n} \sum_{0 \leq j+k \leq n} \left(\frac{1}{3^n} \frac{n!}{j!k!(n-k-j)!} \right)^2. \end{aligned}$$

Use that $\frac{n!}{j!k!(n-j-k)!}3^{-n}$ is the probability that when n balls are placed at random in 3 boxes, the number in each box is j, k, and $n-j-k$. This probability is maximized when $k = j = n/3$ (or as close to $n/3$ as possible). This gives

$$\begin{aligned} P_{00}^{2n} &\leq \frac{1}{2^{2n}} \binom{2n}{n} \left(\frac{1}{3^n} \frac{n!}{(n/3)!(n/3)!(n/3)!} \right) \left(\sum_{0 \leq j+k \leq n} \frac{1}{3^n} \frac{n!}{j!k!(n-j-k)!} \right) \\ &= \frac{1}{2^{2n}} \binom{2n}{n} \left(\frac{1}{3^n} \frac{n!}{(n/3)!(n/3)!(n/3)!} \right). \end{aligned}$$

Application of Stirling's approximation gives $P_{00}^{2n} \leq K/n^{3/2}$ for some positive constant K. Hence,

$$\sum_{n=1}^{\infty} P_{00}^{2n} \leq K \sum_{n=1}^{\infty} \frac{1}{n^{3/2}} < \infty,$$

which gives that the origin is a transient state. Since the random walk is irreducible it follows that the walk is transient.

3.22 Let T be the first return time to state 1, for the chain started in one. a) If the chain first returns to 1 after at least n steps, then the chain first transitioned to 2 and stayed at state 2 for $n-2$ steps. This occurs with probability $p(1-q)^{n-2}$.

b)

$$E(T) = \sum_{n=1}^{\infty} P(T \geq n) = 1 + \sum_{n=2}^{\infty} p(1-q)^{n-2} = 1 + \frac{p}{q} = \frac{p+q}{q} = \frac{1}{\pi_1}.$$

3.23 The chain is finite. It is irreducible, since 1 is accessible from all states, and all states are accessible from 1. It is aperiodic, since $P_{11} > 0$. Thus the chain is ergodic. The vector

$$\boldsymbol{x} = \left(1, \frac{k-1}{k}, \frac{k-2}{k}, \ldots, \frac{1}{k}\right)$$

satisfies $\boldsymbol{xP} = \boldsymbol{x}$. The sum of the entries is

$$\sum_{i=1}^{k} \left(1 - \frac{i-1}{k}\right) = k - \frac{k-1}{2} = \frac{k+1}{2}.$$

Normalizing gives the unique limiting distribution $\pi_i = 2(k+1-i)/(k(k+1))$, for $i = 1, \ldots, k$.

3.24 Show that the stationary distribution for the modified Ehrenfest chain is binomial with parameters N and $1/2$.

The base case $x_0 = 1$ is clear. Suppose the result is true for all $k \leq j$. We need to show the result is true for x_{j+1}. We have that

$$\begin{aligned}
x_{j+1} &= \frac{N}{j+1}\left(x_j - \frac{N-j+1}{N} x_{j-1}\right) \\
&= \frac{N}{j+1}\left(\frac{N!}{j!(N-j)!} - \frac{N-j+1}{N}\frac{N!}{(j-1)!(N-j+1)!}\right) \\
&= \frac{N}{j+1}\left(\frac{N!}{j!(N-j)!} - \frac{(N-1)!}{(j-1)!(N-j)1}\right) \\
&= \frac{N!N}{(j+1)!(N-j)!} - \frac{N!j}{(j+1)!(N-j)!} \\
&= \frac{N!(N-j)}{(j+1)!(N-j)!} = \frac{N!}{(j+1)!(N-j-1)!} = \binom{N}{j+1}.
\end{aligned}$$

3.25 Bernoulli-Laplace model of diffusion.

a) Find the stationary distribution for the cases $k = 2$ and $k = 3$.

$$\boldsymbol{P}2 = \begin{pmatrix} 0 & 1 & 0 \\ 1/4 & 1/2 & 1/4 \\ 0 & 1 & 0 \end{pmatrix},$$

with stationary distribution $\pi = (1/6, 4/6, 1/6)$.

$$\boldsymbol{P}3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1/9 & 4/9 & 4/9 & 0 \\ 0 & 4/9 & 4/9 & 1/9 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

with stationary distribution $\pi = (1/20, 9/20, 9/20, 1/20)$.

b) For the general case, the stationary equations are

$$\begin{aligned}\pi_j &= \pi_{j-1}P_{j-1,j} + \pi_j P_{j,j} + \pi_{j+1}P_{j_1,j} \\ &= \pi_{j-1}\frac{(k-j+1)^2}{k^2} + \pi_j\frac{2j(k-j)}{k^2} + \pi_{j+1}\frac{(j+1)^2}{k^2}.\end{aligned}$$

The equation is satisfied by $\pi_j = \binom{k}{j}^2 / \binom{2k}{k}$.

3.26 The only stationary distribution is $\pi = (0, \ldots, 0, 1)$.

3.27 a) State 0 is accessible from all non-zero states. And every state is accessible from 0. Thus the chain is irreducible. Further from 0, 0 is reachable in k steps for all $k > 1$. Thus, 0 is aperiodic, and hence the chain is aperiodic.

b) From 0, the chain first returns to 0 in exactly $n \geq 2$ steps by moving forward to state $n-1$ and then transitioning to 0, with probability

$$1\left(\frac{1}{2}\right)\cdots\left(\frac{n-2}{n-1}\right)\left(\frac{1}{n}\right) = \frac{1}{(n-1)n}.$$

The probability of eventually returning to 0 is

$$\sum_{n=2}^{\infty}\frac{1}{(n-1)n} = \sum_{n=2}^{\infty}\left(\frac{1}{n-1} - \frac{1}{n}\right) = 1.$$

Thus, 0 is recurrent, and the chain is recurrent.

c) The expected time to return to 0 is

$$\sum_{n=2}^{\infty} n\frac{1}{(n-1)n} = \sum_{n=2}^{\infty}\frac{1}{n-1} = \infty.$$

Thus, the chain is null recurrent.

3.28 $\{1, 2, 3\}$ is a recurrent communication class, and $\{4, 5, 6, 7\}$ is a transient communication class. The submatrix corresponding to the recurrent class is doubly-stochastic and thus the limiting distribution is uniform for these states. The limiting matrix $\lim_{n\to} \boldsymbol{P}^n$ is described as follows: all entries in the first three columns are 1/3; all entries elsewhere are 0.

3.29 The communication classes are: (i) $\{a\}$ transient; (ii) $\{e\}$ recurrent; (iii) $\{c, d\}$ transient; (iv) $\{b, f, g\}$ recurrent, with period 2 (all other states have period 1).

3.30 Suppose the graph is bipartite. Color the vertices accordingly as black or white. From a black vertex, all adjacent vertices are white. From a white vertex all adjacent vertices are black. Thus from a black vertex, the walk can only return to a black vertex in an even number of steps. Similarly for white. Thus the period of the walk is 2 and the walk is periodic.

Suppose the walk is periodic. Since the graph is connected, the chain is irreducible. Given adjacent vertices v and w. observe that $P^2_{v,v} = \sum_k P_{v,k}P_{k,v} \geq P_{v,w}P_{w,v} > 0$. It follows that the period of v, and hence the chain, is either 1 or 2. Thus if the walk is periodic, the period is 2. Starting with any vertex v, the vertex set can be partitioned into two disjoint sets which are accessible from v by either an odd or even number of steps. That is, the graph is bipartite.

3.31 The desired PageRank matrix is

$$(0.9)\begin{pmatrix} 0 & 1/2 & 1/2 & 0 \\ 0 & 0 & 1 & 0 \\ 1/3 & 1/3 & 0 & 1/3 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix} + (0.1)\begin{pmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & 1/4 & 1/4 \end{pmatrix}$$

$$= \begin{pmatrix} 0.025 & 0.475 & 0.475 & 0.025 \\ 0.025 & 0.025 & 0.925 & 0.025 \\ 0.325 & 0.325 & 0.025 & 0.325 \\ 0.25 & 0.25 & 0.25 & 0.25 \end{pmatrix}.$$

The stationary distribution is $\pi = (0.1796, 0.2604, 0.3805, 0.1796)$, which gives the PageRank for the respective nodes a, b, c, d.

3.32 Let $X_0, X_1, \ldots$ be an ergodic Markov chain with transition matrix $\boldsymbol{P}$ and stationary distribution $\boldsymbol{\pi}$. Define the bivariate process $Z_n = (X_n, X_{n-1})$, for $n \geq 1$, with $Z_0 = (X_0, X_0)$.

a) Since the X chain is Markov, the distribution of X_n and X_{n-1} given the past only depends on X_{n-1}.

b)

$$\begin{aligned} P(Z_n = (i,j)|Z_{n-1} = (s,t)) &= P(X_{n-1} = i, X_n = j|X_{n-2} = s, X_{n-1} = t) \\ &= \frac{P(X_n = j, X_{n-1} = t, X_{n-2} = s)}{P(X_{n-1} = t, X_{n-2} = s)} \\ &= \frac{P(X_n = j|X_{n-1} = t, X_{n-2} = s)P(X_{n-1} = t, X_{n-2} = s)}{P(X_{n-1} = t, X_{n-2} = s)} \\ &= P(X_n = j|X_{n-1} = t) = P_{tj}, \text{ if } i = t. \end{aligned}$$

c)

$$\begin{aligned} P(Z_n = (i,j)) = P(X_{n-1} = i, X_n = j) &= P(X_n = j|X_{n-1} = i)P(X_{n-1} = i) \\ &\to P_{ij}\pi_i, \quad \text{as } n \to \infty. \end{aligned}$$

3.33 Suppose $\boldsymbol{P}$ is a stochastic matrix. Show that if $\boldsymbol{P}^N$ is positive, then $\boldsymbol{P}^{N+m}$ is positive for all $m \geq 0$.

We have $P_{ij}^{N+m} = \sum_k P_{ik}^m P_{kj}^N$. Since $\boldsymbol{P}^N$ is positive, $P_{kj}^N > 0$. Thus the only way for P_{ij}^{N+m} to be 0 is for $P_{ik}^m = 0$ for all k. However, since P^m is a stochastic matrix the rows sum to 1 and at least one row entry must be positive. It follows that $P_{ij}^{N+m} > 0$.

3.34 Show that $\widetilde{\boldsymbol{P}}$ is a stochastic matrix for an ergodic Markov chain with the same stationary distribution as $\boldsymbol{P}$. Give an intuitive description for how the $\widetilde{\boldsymbol{P}}$ chain evolves compared to the $\boldsymbol{P}$-chain.

Suppose the original chain is at some state. Flip a coin with heads probability p. If the coin lands heads, transition to a new state according to $\boldsymbol{P}$. If tails, stay put. The chain is aperiodic since $\widetilde{P}_{ii} = pP_{ii} + (1-p) > 0$. Hence, the chain is ergodic. If $\boldsymbol{\pi}$ is the stationary distribution of the $\boldsymbol{P}$-chain, then

$$\boldsymbol{\pi}\widetilde{\boldsymbol{P}} = \boldsymbol{\pi}(p\boldsymbol{P} + (1-p)\boldsymbol{I}) = p\boldsymbol{\pi}\boldsymbol{P} + (1-p)\boldsymbol{\pi}\boldsymbol{I} = p\boldsymbol{\pi} + (1-p)\boldsymbol{\pi} = \boldsymbol{\pi}.$$

3.35 Let $\boldsymbol{Q}$ be a $k \times k$ stochastic matrix. Let $\boldsymbol{A}$ be a $k \times k$ matrix each of whose entries is $1/k$. For $0 < p < 1$, let

$$\boldsymbol{P} = p\boldsymbol{Q} + (1-p)\boldsymbol{A}.$$

We have $P_{ij} = pQ_{ij} + (1-p)\frac{1}{k}$. The matrix is stochastic and regular (all entries are positive). The resulting chain is ergodic.

3.36 Let $X_0, X_1, \ldots$ be an ergodic Markov chain on $\{1, \ldots, k\}$ with stationary distribution $\boldsymbol{\pi}$. Suppose the chain is in stationarity. Find

$$\mathrm{Cov}(X_m, X_{m+n}) \quad \text{and} \quad \lim_{n\to\infty} \mathrm{Cov}(X_m, X_{m+n}).$$

$$\mathrm{Cov}(X_m, X_{m+n}) = E(X_m X_{m+n}) - E(X_m)E(X_{m+n}).$$

Since the chain is in stationarity,

$$E(X_m) = E(X_{m+n}) = \sum_{i=1}^{k} i\pi_i.$$

Also,

$$\begin{aligned} E(X_m X_{m+n}) &= \sum_{i=1}^{k}\sum_{j=1}^{k} ijP(X_m = i, X_{m+n} = j) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} ijP(X_{m+n} = j | X_m = i)P(X_m = i) \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} ijP_{ij}^{n}\pi_i. \end{aligned}$$

This gives

$$\begin{aligned}\operatorname{Cov}(X_m, X_{m+n}) &= \sum_{i=1}^{k}\sum_{j=1}^{k} ijP_{ij}^{n}\pi_i - \left(\sum_{i=1}^{k} i\pi_i\right)^2 \\ &\to \sum_{i=1}^{k}\sum_{j=1}^{k} ij\pi_j\pi_i - \left(\sum_{i=1}^{k} i\pi_i\right)^2 \\ &= \left(\sum_{i=1}^{k} i\pi_i\right)^2 - \left(\sum_{i=1}^{k} i\pi_i\right)^2 = 0, \text{ as } n \to \infty.\end{aligned}$$

3.37 Show that all two-state Markov chains, except for the trivial chain whose transition matrix is the identity matrix, are time-reversible.

We have $\boldsymbol{P} = \begin{pmatrix} 1-p & p \\ q & 1-q \end{pmatrix}$ and $\boldsymbol{\pi} = (q/(p+q), p/(p+q))$. This gives

$$\pi_1 P_{12} = \frac{pq}{p+q} = \pi_2 P_{21}.$$

3.38 You throw five dice and set aside those dice which are sixes. Throw the remaining dice and again set aside the sixes. Continue until you get all sixes.

a) The transition probabilities are

$$P_{ij} = \binom{5-i}{j-i}\left(\frac{1}{6}\right)^{j-i}\left(\frac{5}{6}\right)^{5-j}, \text{ for } 0 \le i, j \le 5.$$

This gives

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\!\!\begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} 0.402 & 0.402 & 0.161 & 0.032 & 0.003 & 0.000 \\ 0 & 0.482 & 0.386 & 0.116 & 0.015 & 0.008 \\ 0 & 0 & 0.579 & 0.347 & 0.069 & 0.005 \\ 0 & 0 & 0 & 0.694 & 0.278 & 0.028 \\ 0 & 0 & 0 & 0 & 0.833 & 0.167 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

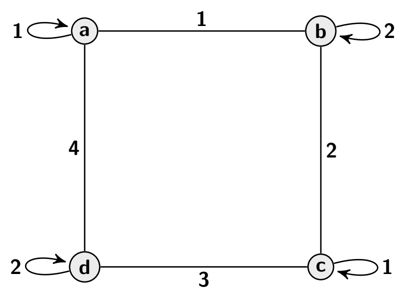
b) The sum of the first row of the fundamental matrix is 13.0237, the desired average number of turns.

3.39 Show that if $X_0, X_1, \ldots$ is reversible, then

$$\begin{aligned}P(X_0 = i_0, X_1 = i_1, \ldots, X_n = i_n) &= \pi_{i_0} P_{i_0,i_1} P_{i_1,i_2} \cdots P_{i_{n-1},i_n} \\ &= \pi_{i_1} P_{i_1,i_0} P_{i_1,i_2} \cdots P_{i_{n-1},i_n} \\ &= P_{i_1,i_0} \pi_{i_2} P_{i_2,i_1} \cdots P_{i_{n-1},i_n} \\ &= \cdots = P_{i_1,i_0} P_{i_2,i_1} \cdots P_{i_n,i_{n-1}} \pi_n \\ &= P(X_0 = i_n, X_1 = i_{n-1}, \ldots, X_{n-1} = i_1, X_n = i_0)\end{aligned}$$

3.40 Consider a *biased random walk* on the n-cycle, which moves one direction with probability p and the other direction with probability $1-p$. The transition matrix is doubly stochastic and the stationary distribution is uniform. Thus the chain is reversible if and only if the transition matrix is symmetric. This occurs if and only if $p = 1 - p$. That is, $p = 1/2$.

3.41 The stationary distribution is $\pi = (3/13, 5/26, 3/13, 9/26)$ and one checks that $\pi_i P_{ij} = \pi_j P_{ji}$ for all i, j. The desired weighted graph is



3.42 Consider random walk on $\{0, 1, 2, \ldots\}$ with one reflecting boundary. If the walk is at 0, it moves to 1 on the next step. Otherwise, it moves left, with probability p, or right, with probability $1 - p$. For what values of p is the chain reversible? For such p, find the stationary distribution.

The chain is reversible for $p > 1/2$. For such p, the stationary distribution is

$$\pi_0 = \frac{p-q}{p-q+1}$$
$$\pi_k = \frac{q^{k-1}(p-q)}{p^k(p-q+1)}, \text{ for } k = 1, 2, \ldots$$

3.43 a) The finite chain is irreducible and aperiodic for all $0 \le p, q \le 1$, except for $p = q = 0$ and $p = q = 1$.

b) The stationary distribution is found to be

$$\boldsymbol{\pi} \left(\frac{p+q}{3}, \frac{1}{6}, \frac{2-p-q}{3}, \frac{1}{6} \right).$$

Checking $\pi_i P_{ij} = \pi_j P_{ji}$, reversibility is satisfied if $p = q$, for all $0 < p < 1$.

3.44 The detailed-balance equations give

$$r\pi_a = q\pi_c, \qquad r\pi_g = q\pi_c, \qquad q\pi_t = r\pi_a$$

from which it follows that

$$(\pi_a, \pi_g, \pi_c, \pi_t) = \frac{1}{2q+2r} (q, q, r, r).$$

For the specific case $p = 0.1$, $q = 0.2$, and $r = 0.3$, this gives $\pi = (2/10, 2/10, 3/10, 3/10)$.

3.45 If $\boldsymbol{P}$ is the transition matrix of a reversible Markov chain, show that $\boldsymbol{P}^2$ is too.

The result follows from the fact that

$$\pi_i P_{ij}^2 = \pi_i \sum_k P_{ik} P_{kj} = \sum_k \pi_k P_{ki} P_{kj} = \sum_k P_{kj} \pi_j P_{jk} = \pi_j P_{ji}^2.$$

3.46 Given a Markov chain with transition matrix $\boldsymbol{P}$ and stationary distribution $\boldsymbol{\pi}$, the *time reversal* is a Markov chain with transition matrix $\widetilde{\boldsymbol{P}}$ defined by

$$\widetilde{P}_{ij} = \frac{\pi_j P_{ji}}{\pi_i}, \text{ for all } i, j.$$

a) The chain is reversible if and only if

$$\widetilde{P}_{ij} = \frac{\pi_j P_{ji}}{\pi_i} = \frac{\pi_i P_{ij}}{\pi_i} = P_{ij}, \text{ for all } i, j.$$

b) Let $\boldsymbol{\pi}$ be the stationary distribution of the original chain. Then

$$(\pi \widetilde{P})_j = \sum_i \pi_i \widetilde{P}_{ij} = \sum_i \pi_i \frac{\pi_j P_{ji}}{\pi_i} = \pi_j.$$

Thus $\boldsymbol{\pi}$ is the stationary distribution of the reversal chain.

3.47 The transition matrix of the time reversal chain is

$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/3 & 4/9 & 2/9 \\ 0 & 1/2 & 1/2 \\ 1/2 & 0 & 1/2 \end{pmatrix} \end{array}.$$

3.48 The transition matrix of the time reversal chain is

$$\boldsymbol{P} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} 1-\alpha & 0 & \alpha \\ \beta & 1-\beta & 0 \\ 0 & \gamma & 1-\gamma \end{pmatrix} \end{array}.$$

3.49 By first-step analysis show that $\boldsymbol{B} = (\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{R}$.

Suppose i is transient and j is absorbing. From i, the chain either (i) moves to j, and is absorbed, (ii) moves to an absorbing state $k \neq j$ and is not absorbed in j, or (iii) moves to a transient state t. This gives

$$B_{ij} = P_{ij} + \sum_{t \in T} P_{it} B_{tj} = R_{ij} + \sum_{t \in T} Q_{it} B_{tj}.$$

In matrix terms, $\boldsymbol{B} = \boldsymbol{R} + \boldsymbol{Q}\boldsymbol{B}$, which gives $(\boldsymbol{I} - \boldsymbol{Q})\boldsymbol{B} = \boldsymbol{R}$, from which the result follows, using the fact that $\boldsymbol{I} - \boldsymbol{Q}$ is invertible.

3.50 Consider the following method for shuffling a deck of cards. Pick two cards from the deck uniformly at random and then switch their positions. If the same two cards are chosen, the deck does not change. This is called the *random transpositions* shuffle.

a) The chain is irreducible. For instance, all states communicate with the identity permutation. The chain is also aperiodic as diagonal entries of the transition matrix are non-zero. The transition matrix is symmetric and hence doubly-stochastic. Thus the stationary distribution is uniform.

b)

$$\boldsymbol{P} = \begin{array}{c} \\ abc \\ acb \\ bac \\ bca \\ cab \\ cba \end{array} \begin{array}{c} \begin{array}{cccccc} abc & acb & bac & bca & cab & cba \end{array} \\ \begin{pmatrix} 1/3 & 2/9 & 2/9 & 0 & 0 & 2/9 \\ 2/9 & 1/3 & 0 & 2/9 & 2/9 & 0 \\ 2/9 & 0 & 1/3 & 2/9 & 2/9 & 0 \\ 0 & 2/9 & 2/9 & 1/3 & 0 & 2/9 \\ 0 & 2/9 & 2/9 & 0 & 1/3 & 2/9 \\ 2/9 & 0 & 0 & 2/9 & 2/9 & 1/3 \end{pmatrix} \end{array}.$$

c) Modify the chain so *cba* is an absorbing state. Then

$$(\boldsymbol{I} - \boldsymbol{Q})^{-1} = \begin{pmatrix} 5/2 & 3/2 & 3/2 & 1 & 1 \\ 3/2 & 3 & 3/2 & 3/2 & 3/2 \\ 3/2 & 3/2 & 3 & 3/2 & 3/2 \\ 1 & 3/2 & 3/2 & 5/2 & 1 \\ 1 & 3/2 & 3/2 & 1/ & 5/2 \end{pmatrix}.$$

The sum of the first row is 7.5, which is the desired expectation.

3.51 a)

$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \end{array} \\ \begin{pmatrix} 1/6 & 1/6 & 1/6 & 1/6 & 1/6 & 1/6 \\ 5/6 & 1/6 & 0 & 0 & 0 & 0 \\ 0 & 4/6 & 2/6 & 0 & 0 & 0 \\ 0 & 0 & 3/6 & 3/6 & 0 & 0 \\ 0 & 0 & 02/6 & 4/6 & 0 & \\ 0 & 0 & 0 & 0 & 1/6 & 5/6 \end{pmatrix} \end{array}.$$

The matrix is doubly-stochastic and the stationary distribution is uniform. The expected return time is $1/(1/6) = 6$.

b) Make state 1 an absorbing state. The fundamental matrix is

$$(\boldsymbol{I} - \boldsymbol{Q})^{-1} = \begin{pmatrix} 6/5 & 0 & 0 & 0 & 0 \\ 6/5 & 3/2 & 0 & 0 & 0 \\ 6/5 & 3/2 & 2 & 0 & 0 \\ 6/5 & 3/2 & 2 & 3 & 0 \\ 6/5 & 3/2 & 2 & 3 & 6 \end{pmatrix}.$$

The sum of the last row is 13.7, which is the desired expectation.

b) Find the expected number of shuffles for the bottom card to reach the top of the deck.

3.52 a) The transition matrix for the game is

$$
\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\begin{pmatrix}
0 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 & 1/4 & 0 & 0 \\
0 & 0 & 0 & 1/2 & 1/4 & 0 & 0 & 1/4 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 \\
0 & 0 & 0 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/4 & 0 & 1/4 & 1/4 & 0 & 1/4 \\
0 & 0 & 0 & 0 & 1/4 & 0 & 0 & 1/2 & 0 & 1/4 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array}.
$$

The sum of the first row of the fundamental matrix gives the expected duration of the game as 8.625 steps.

b) Make squares 3 and 9 absorbing states. This gives

$$
(\boldsymbol{I}-\boldsymbol{Q})^{-1}\boldsymbol{R} =
\begin{array}{c}
\begin{array}{cccccccc} 0 & 1 & 2 & 4 & 5 & 6 & 7 & 8 \end{array} \\
\begin{pmatrix}
1 & 1/4 & 0 & 15/16 & 0 & 5/16 & 5/4 & 0 \\
0 & 1 & 0 & 3/4 & 0 & 1/4 & 1 & 0 \\
0 & 0 & 1 & 1 & 0 & 1/3 & 8/3 & 0 \\
0 & 0 & 0 & 2 & 0 & 2/3 & 4/3 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 5/3 & 4/370 & \\
0 & 0 & 0 & 1 & 0 & 1/3 & 8/3 & 0 \\
0 & 0 & 0 & 2 & 0 & 2/3 & 4/3 & 1
\end{pmatrix}
\end{array}
\begin{array}{c}
\begin{array}{cc} 3 & 9 \end{array} \\
\begin{pmatrix}
1/4 & 0 \\
1/2 & 0 \\
0 & 0 \\
1/4 & 0 \\
1 & 0 \\
0 & 1/4 \\
0 & 1/4 \\
0 & 0
\end{pmatrix}
\end{array}
$$

$$
= \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array}
\begin{array}{c}
\begin{array}{cc} 3 & 9 \end{array} \\
\begin{pmatrix}
39/64 & 25/64 \\
11/16 & 5/16 \\
1/4 & 3/4 \\
1/2 & 1/2 \\
1 & 0 \\
1/4 & 3/4 \\
1/4 & 3/4 \\
1/2 & 1/2
\end{pmatrix}
\end{array}.
$$

From square 6, the probability of of hitting 3 before 9 is 1/4.

3.53 a)

$$
(\boldsymbol{I}-\boldsymbol{Q})^{-1}\boldsymbol{R} = \begin{pmatrix} 1 & p-1 \\ p-1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} p & 0 \\ 0 & p \end{pmatrix} = \begin{pmatrix} 1/(2-p) & (1-p)/(2-p) \\ (1-p)/(2-p) & 1/(2-p) \end{pmatrix}.
$$

The desired probability is $1/(2-p)$.

b) Denote state Axy to mean that A has possession and x points, and B has y points. Similarly define Bxy. The transition matrix is

$$
\boldsymbol{P} = \begin{array}{c} \\ A00 \\ A30 \\ A33 \\ A03 \\ B00 \\ B30 \\ B33 \\ B03 \\ A \\ B \end{array}
\begin{array}{c}
\begin{array}{cccccccccc} A00 & A30 & A33 & A03 & B00 & B30 & B33 & B03 & A & B \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 1-p & \beta & 0 & 0 & \alpha & 0 \\
0 & 0 & 0 & 0 & 0 & 1-p & 0 & 0 & p & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1-p & 0 & p & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & \beta & 1-p & \alpha & 0 \\
1-p & 0 & 0 & \beta & 0 & 0 & 0 & 0 & 0 & \alpha \\
0 & 1-p & \beta & 0 & 0 & 0 & 0 & 0 & 0 & \alpha \\
0 & 0 & 1-p & 0 & 0 & 0 & 0 & 0 & 0 & p \\
0 & 0 & 0 & 1-p & 0 & 0 & 0 & 0 & 0 & p \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\end{array},
$$

where $p = \alpha + \beta$.

c) For the first overtime rule, $p = (1270+737)/6049$ and team A wins with probability $1/(2-p) \approx 0.599$. For the second overtime rule, let $\alpha = 1270/6049$ and $\beta = 737/6049$. Compute $(\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{R}$. The $(1,1)$ entry of the resulting matrix gives the desired probability 0.565.

3.54 a) How many rooms, on average, will the mouse visit before it finds the cheese?

The fundamental matrix is

$$
\left(\boldsymbol{I}_8 - \begin{pmatrix}
0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 & 0 \\
1/3 & 0 & 1/3 & 0 & 1/3 & 0 & 0 & 0 \\
0 & 1/2 & 0 & 0 & 0 & 1/2 & 0 & 0 \\
1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\
0 & 1/3 & 0 & 0 & 0 & 1/3 & 0 & 1/3 \\
0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0
\end{pmatrix} \right)^{-1}
$$

$$
= \begin{pmatrix}
15/2 & 33/4 & 5 & 15/2 & 6 & 9/2 & 15/4 & 2 \\
11/2 & 33/4 & 5 & 11/2 & 6 & 9/2 & 11/4 & 2 \\
5 & 15/2 & 6 & 5 & 6 & 5 & 5/2 & 2 \\
15/2 & 33/4 & 5 & 19/2 & 6 & 9/2 & 19/4 & 2 \\
4 & 6 & 4 & 4 & 6 & 4 & 2 & 2 \\
9/2 & 27/4 & 5 & 9/2 & 6 & 11/2 & 9/4 & 2 \\
15/2 & 33/4 & 5 & 19/2 & 6 & 9/2 & 23/4 & 2 \\
2 & 3 & 2 & 2 & 3 & 2 & 1 & 2
\end{pmatrix}.
$$

The sum of the first row is the desired expectation 44.5.

b) How many times, on average, will the mouse revisit room A before it finds the cheese? The (1,1) entry of the above matrix gives the desired average of 7.5 visits.

3.55 In a sequence of fair coin flips, how many flips, on average, are required to first see the pattern H-H-T-H?

Let

$$
\boldsymbol{P} = \begin{array}{c} \\ \emptyset \\ H \\ HH \\ HHT \\ HHTH \end{array} \begin{array}{c} \begin{array}{ccccc} \emptyset & H & HH & HHT & HHTH \end{array} \\ \begin{pmatrix} 1/2 & 1/2 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 & 0 \\ 1/2 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.
$$

The row sum of the fundamental matrix is 18, which is the desired average.

3.56 A biased coin has heads probability 1/3 and tails probability 2/3. If the coin is tossed repeatedly, find the expected number of flips required until the pattern H-T-T-H-H appears.

Let

$$
\boldsymbol{P} = \begin{array}{c} \\ \emptyset \\ H \\ HT \\ HTT \\ HTTH \\ HTTHH \end{array} \begin{array}{c} \begin{array}{cccccc} \emptyset & H & HT & HTT & HTTH & HTTHH \end{array} \\ \begin{pmatrix} 2/3 & 1/3 & 0 & 0 & 0 & 0 \\ 0 & 1/3 & 2/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 2/3 & 0 & 0 \\ 2/3 & 0 & 0 & 0 & 1/3 & 0 \\ 2/3 & 0 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.
$$

The row sum of the fundamental matrix is $291/4 = 72.75$, which is the desired average.

3.57 Average number of flips for the following patterns are

HHH, TTT: 14

HHT, TTH, HTT, THH: 8

HTH, THT: 6

3.58 Construct a *pattern* Markov chain with transition matrix

$$
\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 0 \\ 00 \\ 001 \\ 0011 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 0 & 00 & 001 & 0011 \end{array} \\ \begin{pmatrix} 1/4 & 3/4 & 0 & 0 & 0 \\ 3/4 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 1/4 & 3/4 & 0 \\ 0 & 3/4 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.
$$

The fundamental matrix is

$$
(\boldsymbol{I} - \boldsymbol{Q})^{-1} = \begin{array}{c} \\ 1 \\ 0 \\ 00 \\ 001 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 0 & 00 & 001 \end{array} \\ \begin{pmatrix} 52/3 & 16 & 16/3 & 4 \\ 16 & 16 & 16/3 & 4 \\ 12 & 12 & 16/3 & 4 \\ 12 & 12 & 4 & 4 \end{pmatrix} \end{array}
$$

The first row sum is $52/3 + 16 + 16/3 + 4 = 42.667$, which is the average number of steps in the *pattern* chain until absorption if the chain starts in the one-outcome pattern represented by "1". Similarly, the second row sum is 41.333, which is the average number of steps in the pattern chain until absorption if the chain starts in the one-outcome pattern represented by "0." These represent realizations where one outcome has already occurred. When we start our trials no patterns have occurred yet. Thus, conditioning on the first outcome gives the desired expectation $(1/2)(42.667+1)+(1/2)(41.333+1)=43$ steps.

This problem could also be modeled with a 6-state chain which includes the initial *null* state $\emptyset$, using the transition matrix

$$\boldsymbol{P} = \begin{array}{c} \\ \emptyset \\ 1 \\ 0 \\ 00 \\ 001 \\ 0011 \end{array} \begin{array}{c} \begin{array}{cccccc} \emptyset & 1 & 0 & 00 & 001 & 0011 \end{array} \\ \begin{pmatrix} 0 & 1/2 & 1/2 & 0 & 0 & 0 \\ 0 & 1/4 & 3/4 & 0 & 0 & 0 \\ 0 & 3/4 & 0 & 1/4 & 0 & 0 \\ 0 & 0 & 0 & 1/4 & 3/4 & 0 \\ 0 & 0 & 3/4 & 0 & 0 & 1/4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}.$$

3.59 a) Stationary distribution is

$$(\pi_a, \pi_b, \pi_c, \pi_d) = (6/23, 6/23, 5/23, 6/23).$$

Expected number of steps to return to a is $1/\pi_a = 23/6 = 3.83$.

b) Make b an absorbing state. The fundamental matrix is

$$\left(\boldsymbol{I}_3 - \begin{pmatrix} 1/6 & 0 & 4/6 \\ 0 & 2/5 & 1/5 \\ 4/6 & 1/6 & 1/6 \end{pmatrix} \right) = \begin{pmatrix} 42/11 & 10/11 & 36/11 \\ 12/11 & 45/11 & 15/11 \\ 36/11 & 25/22 & 45/11 \end{pmatrix}.$$

The desired expectation is $42/11 + 10/11 + 36/11 = 8$.

c) Make both b and c absorbing states. Then

$$(\boldsymbol{I} - \boldsymbol{Q})^{-1}\boldsymbol{R} = \begin{array}{c} \\ a \\ d \end{array} \begin{array}{c} \begin{array}{cc} b & c \end{array} \\ \begin{pmatrix} 5/9 & 4/9 \\ 4/9 & 5/9 \end{pmatrix} \end{array}.$$

The desired expectation is 5/9.

3.60 For a Markov chain started in state i, let T denote the *fifth time* the chain visits state i. Is T a stopping time?

Yes. For each n, whether or not the event $\{T = n\}$ occurs can be determined from $X_0, \ldots, X_n$.

3.61 Weather chain. Yes. T is a stopping time. For each n, whether or not the event $\{T = n\}$ occurs can be determined from $X_0, \ldots, X_n$.

3.62 A constant is trivially a stopping time. From the strong Markov property it follows that for any constant S, $X_S, X_{S+1}, X_{S+2}, \ldots$ is a Markov chain.

3.63 `R`:

$$\boldsymbol{\pi} = (0.325, 0.207, 0.304, 0.132, 0.030, .003, .0003).$$

3.64 R: Forest ecosystems. Making the 12th state absorbing, the sum of the first row of the fundamental matrix is 10,821.68. If state changes represent five-year intervals, the model gives that it takes, on average, 5(10821.68) = 54,108.4 years.

3.65 R: Gambler's ruin. Probability of ruin is 0.65.

3.66 R: Simulate the expected hitting time for the random walk on the hexagon in Exercise 3.19.

```
 simlist <- numeric(100000)
for (i in 1:100000) {
flag <- 0
ct <- 0
s <- 1
while (flag == 0) {
if (s==1) ns <- sample(c(2,3,5,6),1)
if (s == 2) ns <- sample(c(1,3),1)
if (s == 3) ns <- sample(c(2,1,4,5),1)
if (s == 5) ns <- sample(c(1,3,4,6),1)
if (s == 6) ns <- sample(c(1,5),1)
ct <- ct + 1
if (ns == 4) flag <- 1
s <- ns
}
simlist[i] <- ct
}
mean(simlist)
```

3.67 R: Simulate the dice game. Verify numerically the theoretical expectation for the number of throws needed to get all sixes.

```
 > allsixes <- function() {
+ i <- 0
+ ct <- 0
+ while (ct < 5)
+ { x <- sample(1:6,5-ct,replace=T)
+ sixes <- sum(x==6)
+ ct <- ct + sixes
+ i <- i+1
+ }
+ i
+ }
>
> sim <- replicate(10000, allsixes())
> mean(sim)
[1] 13.0873
```

3.68 R: Write a function `reversal(mat)`, whose input is the transition matrix of an irreducible Markov chain and whose output is the transition matrix of the reversal chain.

```
reversal <- function(mat) {
r <- dim(mat)[1]
c <- dim(mat)[2]
st <- eigen(t(mat))$vectors[,1]
stat <- as.double(st/sum(st))
rmat <- matrix(0,nrow=r,ncol=c)
for (i in 1:r) {
for (j in 1:c) {
rmat[i,j] <- stat[j]*mat[j,i]/stat[i] }
}
return(rmat)   }
```

Chapter 4

4.1 Consider a branching process with offspring distribution $\boldsymbol{a} = (a, b, c)$, where $a+b+c = 1$. Let $\boldsymbol{P}$ be the Markov transition matrix. Exhibit the first three rows of $\boldsymbol{P}$.

First row: $P_{0,j} = 1$, if $j = 0$, and 0, otherwise.

Second row:

$$P_{1,j} = \begin{cases} a, & \text{if } j = 0 \\ b, & \text{if } j = 1 \\ c, & \text{if } j = 2. \end{cases}$$

Third row:

$$\begin{matrix} & 0 & 1 & 2 & 3 & 4 & 5 & \cdots \\ 2 & \big(a^2 & 2ab & 2ac + b^2 & 2bc & c^2 & 0 & \cdots \big) \end{matrix}$$

4.2 Find the probability generating function of a Poisson random variable with parameter λ. Use the pgf to find the mean and variance of the Poisson distribution.

$$G(s) = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda}\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(s\lambda)^k}{k!} = e^{-\lambda} e^{s\lambda} = e^{\lambda(s-1)}.$$

We have $G'(s) = \lambda e^{\lambda(s-1)}$ and $G''(s) = \lambda^2 e^{\lambda(s-1)}$, with $\mu = G'(1) = \lambda$ and $\sigma^2 = \lambda$.

4.3 Let $X \sim \text{Poisson}(\lambda)$ and $Y \sim \text{Poisson}(\mu)$. Suppose X and Y are independent. Use probability generating functions to find the distribution of $X + Y$.

$G_X(s) = e^{\lambda(s-1)}$ and $G_Y(s) = e^{\mu(s-1)}$. Then,

$$G_{X+Y}(s) = G_X(s)G_Y(s) = e^{\lambda(s-1)}e^{\mu(s-1)} = e^{(\lambda+\mu)(s-1)}.$$

Thus, $X + Y$ has a Poisson distribution with parameter $\lambda + \mu$.

4.4 If X is a negative binomial distribution with parameters r and p, then X can be written as the sum of r i.i.d. geometric random variables with parameter p. Use this fact to find the pgf of X. Then use the pgf to find the mean and variance of the negative binomial distribution.

The generating function of the geometric distribution with parameter p is

$$G_X(t) = \frac{pt}{1 - (1-p)t}.$$

This gives the generating function of the negative binomial distribution with parameters r and p

$$G(t) = \left(\frac{pt}{1 - (1-p)t} \right)^r.$$

Relevant derivatives are

$$G'(1) = \frac{r}{p} \quad \text{and} \quad G''(1) = \frac{r(1 - 2p + r)}{p^2}.$$

This gives

$$\mu = G'(1) = \frac{r}{p} \quad \text{and } \sigma^2 = G''(1) + G'(1) - G'(1)^2 = \frac{r(1-p)}{p^2}.$$

4.5 a) The kth factorial moment is $G^{(k)}(1)$.

b) For the binomial distribution,

$$E(X(X-1)\cdots(X-k+1)) = \left.\frac{d^k}{ds^k}\right|_{s=1} ((1-p)+ps)^n = \frac{n!}{(n-k)!}p^k.$$

4.6 a)

$$\begin{aligned}
G_Z(s) = E(s^Z) = E\left(s^{\sum_{i=1}^N X_i}\right) &= \sum_{n=0}^{\infty} E\left(s^{\sum_{i=1}^N X_i} | N=n\right) P(N=n) \\
&= \sum_{n=0}^{\infty} E\left(s^{\sum_{i=1}^n X_i}\right) P(N=n) \\
&= \sum_{n=0}^{\infty} (1-p+ps)^n \frac{e^{-\lambda}\lambda^n}{n!} = e^{-\lambda} \sum_{n=0}^{\infty} \frac{(\lambda(1-p+ps))^n}{n!} \\
&= e^{-\lambda} e^{\lambda(1-p+ps)} = e^{-\lambda p(1-s)}.
\end{aligned}$$

b) The generating function is that of a Poisson random variable with parameter λp, which gives the distribution of Z.

4.7 Give the probability generating function for an offspring distribution in which an individual either dies, with probability $1-p$, or gives birth to three children, with probability p, Also find the mean and variance for the number of children in the fourth generation.

$$\begin{aligned}
G(s) &= (1-p) + ps^3. \\
\mu &= G'(1) = 3p. \\
\sigma^2 &= G''(1) + G'(1) - G'(1)^2 = 6p + 3p - (3p)^2 = 9p(1-p). \\
E(Z^4) &= \mu^4 = (3p)^4 = 81p^4. \\
Var(Z^4) &= 9p(1-p)(3p)^3(81p^4-1)/(3p-1).
\end{aligned}$$

4.8 If X is a discrete random variable with generating function G. Show that

$$P(X \text{ is even}) = \frac{1+G(-1)}{2}.$$

$$G(-1) = \sum_{k=0}^{\infty} (-1)^k P(X=k) = P(X \text{ is even}) - P(X \text{ is odd}).$$

Also,

$$1 = P(X \text{ is even}) + P(X \text{ is odd}).$$

Solving the system gives the result.

4.9 Let $Z_0, Z_1, \ldots$ be a branching process whose offspring distribution mean is μ. Let $Y_n = Z_n/\mu^n$, for $n \neq 0$.

$$\begin{aligned} E(Y_{n+1}|Y_n = k) &= \frac{1}{\mu^{n+1}} E\left(Z_{n+1}|Z_n = k\mu^n\right) = \frac{1}{\mu^{n+1}} E\left(\sum_{i=1}^{k\mu^n} X_i | Z_n = k\mu^n\right) \\ &= \frac{1}{\mu^{n+1}} \sum_{i=1}^{k\mu^n} E(X_i) = \frac{1}{\mu^{n+1}} k\mu^n(\mu) = k, \end{aligned}$$

which gives $E(Y_{n+1}|Y_n) = Y_n$.

4.10 Show by induction that for $\mu \neq 1$,

$$Var(Z_n) = \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1}. \tag{1}$$

It is shown in Section 4.2 that

$$Var(Z_n) = \mu^2 Var(Z_{n-1}) + \sigma^2 \mu^{n-1}, \text{ for } n \geq 1.$$

For $n = 1$, this gives $Var(Z_1) = \sigma^2$. Assuming Equation (1) true for all $k < n$, we have

$$\begin{aligned} Var(Z_n) &= \mu^2 \left(\sigma^2 \mu^{n-2} \frac{\mu^{n-1} - 1}{\mu - 1}\right) + \sigma^2 \mu^{n-1} \\ &= \sigma^2 \mu^{n-1} \left(1 + \frac{\mu^n - \mu}{\mu - 1}\right) \\ &= \sigma^2 \mu^{n-1} \frac{\mu^n - 1}{\mu - 1}. \end{aligned}$$

4.11 Use the generating function representation of Z_n to find $E(Z_n)$.

$$\begin{aligned} E(Z_n) &= G_n'(1) = G'(G_{n-1}(1))G_{n-1}'(1) = G'(1)G_{n-1}'(1) = \mu G_{n-1}'(1) \\ &= \mu^2 G_{n-2}'(1) = \cdots = \mu^{n-1} G_1'(1) = \mu^{n-1} G'(1) = \mu^n. \end{aligned}$$

4.12 A branching process has offspring distribution $\boldsymbol{a} = (1/4, 1/4, 1/2)$.

a) $\mu = 0(1/4) + 1(1/4) + 2(1/2) = 5/4$

b) $G(s) = 1/4 + s/4 + s^2/2$

c) Solve $s = 1/4 + s/4 + s^2/2$ to find extinction probability is $e = 1/2$

d)

$$
\begin{aligned}
G_2(s) = G(G(s)) &= \frac{1}{4} + \frac{G(s)}{4} + \frac{G(s)^2}{2} \\
&= \frac{1}{4} + \frac{1}{4}\left(\frac{1}{4} + \frac{s}{4} + \frac{s^2}{2}\right) + \frac{1}{2}\left(\frac{1}{4} + \frac{s}{4} + \frac{s^2}{4}\right)^2 \\
&= \frac{1}{32}\left(11 + 4s + 9s^2 + 4s^3 + 4s^4\right).
\end{aligned}
$$

e) $P(Z_2 = 0) = G(G(0)) = G(1/4) = 1/4 + 1/16 + 1/32 = 11/32$.

4.13 Use numerical methods to find the extinction probability for a Poisson offspring distribution with parameter $\lambda = 1.5$.

Solving $s = e^{1.5(s-1)}$ gives $e = 0.417$.

4.14 A branching process has offspring distribution $a_0 = p$, $a_1 = 1 - p - q$, and $a_2 = q$.

The mean of the offspring distribution is $\mu = 1 - p - q + 2q = 1 - p + q$. The process is supercritical if $1 - p + q > 1$. That is, $q > p$.

For the extinction probability, solve $s = p + (1-p-q)s + qs^2$, or $qs^2 - (p+q)s + p = 0$. If $q > p$, the roots are 1 and p/q. The latter gives the extinction probability.

4.15 Suppose the offspring distribution is uniform on $\{0, 1, 2, 3, 4\}$. Find the extinction probability.

Solve $s = 1/4 + s/4 + s^2/4 + s^3/4$ to obtain $e = 0.414$.

4.16 Consider a branching process where $Z_0 = k$. Let $G(s)$ be the probability generating function of the offspring distribution. Let $G_n(s)$ be the probability generating function of Z_n for $n = 0, 1, \ldots$

a)

$$
G_1(s) = E\left(s^{Z_1}\right) = E\left(s^{\sum_{i=1}^k X_i}\right) = \prod_{i=1}^k E\left(s^{X_i}\right) = [G(s)]^k.
$$

b) True or False: $G_{n+1}(s) = G_n(G(s))$, for $n = 1, 2, \ldots$

TRUE.

$$
\begin{aligned}
G_{n+1}(s) &= E\left(s^{Z_{n+1}}\right) = E\left(E\left(s^{Z_{n+1}} | Z_n\right)\right) \\
&= E\left(E\left(s^{\sum_{i=1}^{Z_n} X_i} | Z_n\right)\right) = E\left(G(s)^{Z_n}\right) = G_n(G(s)).
\end{aligned}
$$

c) True or False: $G_{n+1}(s) = G(G_n(s))$, for $n = 1, 2, \ldots$

FALSE. For instance, $G_2(s) = G_1(G(s)) = [G(G(s))]^k$. But $G(G_1(s)) = G(G(s)^k)$.

4.17 Let $\boldsymbol{a} = ((1-p), 0, p)$ be the offspring distribution. Suppose the process starts with two individuals.

a)

$$e = \begin{cases} (1-p)/p, & \text{if } p > 1/2 \\ 1, & \text{if } p \leq 1/2. \end{cases}$$

b) In one generation, to change from $2i$ to $2j$ individuals means that k of the $2i$ individuals had 2 offspring, and the remaining $2i - k$ individuals had no offspring. Thus $2k = 2j$, or $k = j$. This gives the one-step transition probabilities

$$P_{2i,2j} = \binom{2i}{j} p^j (1-p)^{2i-j}.$$

4.18 Consider a branching process with offspring distribution

$$\boldsymbol{a} = \left(p^2, 2p(1-p), (1-p)^2\right), \quad \text{for } 0 < p < 1.$$

Extinction probability is $e = p^2/(1-p)^2$.

4.19 Let $T = \min\{n : Z_n = 0\}$. Show that $P(T = n) = G_n(0) - G_{n-1}(0)$.

Extinction first happens at generation n if and only if $Z_n = 0$ and $Z_{n-1} > 0$. Thus,

$$P(T = n) = P(Z_n = 0, Z_{n-1} > 0).$$

Also,

$$\begin{aligned} P(Z_n = 0) &= P(Z_n = 0, Z_{n-1} = 0) + P(Z_n = 0, Z_{n-1} > 0) \\ &= P(Z_{n-1} = 0) + P(Z_n = 0, Z_{n-1} > 0), \end{aligned}$$

since $\{Z_{n-1} = 0\} \subseteq \{Z_n = 0\}$. Rearranging gives

$$P(T = n) = P(Z_n = 0) - P(Z_{n-1} = 0) = G_n(0) - G_{n-1}(0).$$

4.20 Consider an offspring distribution that is geometric with parameter $p = 1/2$.

a) The mean of the offspring distribution is 1. This is the critical case, and the extinction probability is 1.

b) Generating function is

$$G(s) = \sum_{k=0}^{\infty} s^k \left(\frac{1}{2}\right)^{k+1} = \frac{1}{2-s}.$$

This gives $G_1(s) = G(s)$. By the induction hypothesis,

$$\begin{aligned} G_n(s) = G_{n-1}(G(s)) &= \frac{n-1-(n-2)G(s)}{n-1+1-(n-1)G(s)} \\ &= \frac{n-1-(n-2)/(2-s)}{n-(n-1)/(2-s)} = \frac{n-(n-1)s}{n+1-ns}. \end{aligned}$$

c) Find the distribution of the time of extinction.

$$P(T = n) = G_n(0) - G_{n-1}(0) = \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)}, \quad \text{for } n \geq 1.$$

4.21 The *linear fractional case* is one of the few examples in which the generating function $G_n(s)$ can be explicitly computed. Let

$$a_0 = \frac{1-c-p}{1-p}, \quad a_k = cp^{k-1}, \text{ for } k = 1, 2, \ldots, \text{ with } 0 < p < 1.$$

The offspring distribution is a geometric distribution rescaled at 0.

a) $\mu = \sum_{k=0}^{\infty} k a_k = c/(1-p)^2$.

b) If $\mu = 1$, then $c = (1-p)^2$. The offspring generation function is

$$G(s) = \frac{1-(1-p)^2-p}{1-p} + \sum_{k=1}^{\infty}(1-p)^2 s^k p^{k-1} = \frac{(p+(1-2p)s}{1-ps}.$$

By the induction hypothesis,

$$G_{n+1}(s) = G_n(G(s)) = \frac{np-(np+p-1)G(s)}{1-p+np-npG(s)} = \cdots = \frac{(n+1)p-((n+1)p+p-1)s}{1-p+(n+1)p-(n+1)ps}.$$

c) We have $c = 0.2126$, $p = 0.5893$, $\mu = 1.26042$, $e = 0.818509$,and

$$G_n(0) = \frac{e(1-\mu^n)}{e-\mu^n}.$$

This gives $P(Z_3 = 0) = G_3(0) = 0.693$ and $P(Z_{10} = 0) = G_{10}(0) = 0.803$.

4.22 *Linear fractional case, continued.*

a) The offspring distribution is given by $a_0 = 1/2$, $a_k = (1/12)(5/6)^{k-1}$, for $k \geq 1$, with mean $\mu = 3$. This gives $E(Z_4) = \mu^4 = 3^4 = 81$.

b, c) This is the linear fractional case with $c = 1/12$ and $p = 5/6$. The extinction probability is $e = (1-c-p)/(p(1-p)) = 3/5$. Also,

$$P(T=4) = G_4(0) - G_3(0) = \frac{e(1-\mu^n)}{e-\mu^n} - \frac{e(1-\mu^{n-1})}{e-\mu^{n-1}} = 0.006.$$

4.23 Find the generating function of Z in terms of G.

$$\begin{aligned} G_Z(s) = E\left(s^Z\right) &= \sum_{k=0}^{\infty} s^k P(Z=k) = \sum_{k=0}^{\infty} s^k P(X=k|X>0) \\ &= \sum_{k=0}^{\infty} s^k \frac{P(X=k, X>0)}{P(X>0)} = \sum_{k=1}^{\infty} s^k \frac{P(X=k)}{P(X=0)} = \frac{1}{a_0}\left(G(s)-a_0\right) \end{aligned}$$

4.24

$$E(T_n) = E\left(\sum_{k=0}^{n} Z_k\right) = \sum_{k=0}^{\infty} E(Z_k) = \sum_{k=0}^{n} \mu^k = \begin{cases} n+1, & \text{if } \mu = 1 \\ (1-\mu^{n+1})/(1-\mu), & \text{if } \mu \neq 1. \end{cases}$$

It follows that

$$E(T) = \begin{cases} 1/(1-\mu), & \text{if } \mu < 1 \\ +\infty, & \text{if } \mu \geq 1. \end{cases}$$

4.25 *Total progeny, continued.* Let $\phi_n(s) = E(s^{T_n})$ be the probability generating function of T_n.

a) We have $\phi_n(s) = E\left(s^{T_n}|X_0 = 1\right)$. This gives $E\left(s^{T_n}|X_0 = x\right) = [\phi_n(s)]^x$, since each individual of the first generation generates an independent number of descendants. So the pgf for their sum is the product of the pgf's associated with each first generation ancestor. With $Z_0 = 1$,

$$\begin{aligned} \phi_n(s) &= E\left(s^{T_n}\right) = E\left(E\left(s^{T_n}|Z_1\right)\right) \\ &= E\left(sE\left(s^{Z_1+\cdots+Z_n}|Z_1\right)\right) \\ &= sE\left([\phi_{n-1}(s)]^{Z_1}\right) = sG_{Z_1}\left(\phi_{n-1}(s)\right) = sG\left(\phi_{n-1}(s)\right). \end{aligned}$$

The third equality is by the Markov property and because $(Z_1, Z_2, \ldots)$ has the same probabilistic properties as a Galton-Watson branching process conditioned to start with Z_1 individuals.

b) Taking limits, we let $T = \lim_{n\to\infty} \sum_{k=0}^{n} T_k$ and $\phi(s) = \lim_{n\to\infty} \phi_{n-1}(s)$. Appealing to the continuity of the generating function

$$\phi(s) = \lim_{n\to\infty} sG(\phi_{n-1}(s)) = sG(\lim_{n\to\infty} \phi_{n-1}(s)) = sG(\phi(s)).$$

A technical condition to consider is the possibility that $P(T = \infty) > 0$. We do not treat this issue in the text, but note that for such random variables, the probability generating function $E(s^T)$ can still be defined as $s^\infty = 0$, for $0 \leq s < 1$.

c) By the chain rule,

$$\phi'(s) = sG'(\phi(s))\phi'(s) + G(\phi(s)).$$

This gives

$$E(T) = \phi'(1) = G'(\phi(1))\phi'(1) + G(\phi(1)) = G'(1)\phi'(1) + G(1) = \mu\phi'(1) + 1,$$

using that $1 = G(1) = \phi(1)$. Solving for $\phi'(1)$ gives $E(T) = 1/(1-\mu)$.

$$E(T_n) = \sum_{k=0}^{n} E(Z_k) = \sum_{k=0}^{n} \mu^k = \begin{cases} \frac{1-\mu^{n+1}}{1-\mu}, & \text{if } \mu \neq 1 \\ n+1, & \text{if } \mu = 1. \end{cases}$$

This gives

$$E(T) = E(\lim_{n\to\infty} T_n) = \lim_{n\to\infty} E(T_n) = \begin{cases} 1/(1-\mu), & \text{if } \mu < 1 \\ +\infty, & \text{if } \mu \geq 1. \end{cases}$$

4.26 Lottery game.

a)

Net winnings ($)	-1	2	14	999
Probability	$\binom{97}{3}/\binom{100}{3}$	$\binom{3}{1}\binom{97}{2}/\binom{100}{3}$	$\binom{3}{2}\binom{97}{1}/\binom{100}{3}$	$1/\binom{100}{3}$

Expected value is

$$\frac{1}{\binom{100}{3}}\left((-1)\binom{97}{3}+(2)\binom{3}{1}\binom{97}{2}+(14)\binom{3}{2}\binom{97}{1}+999\right)=\frac{10403}{14700}=-0.708.$$

b) The offspring distribution is given by

$$a_3=3\binom{97}{2}/\binom{100}{3}=0.0864, \quad a_{15}=(3)(97)/\binom{100}{3}=0.0018,$$

and $a_0 = 1 - a_3 - a_{15} = 0.9118$, with mean $\mu = 3a_3 + 15a_{15} = 0.286$.

c) We have $G_1(s) = G(s) = 0.9118 + 0.0864s^3 + 0.0018s^{15}$. Numerical evaluation gives $G_2(0) = G_1(G(0)) = G_1(0.9118) = 0.9777$, $G_3(0) = G(G_2(0)) = G(0.9777) = 0.9938$, and $G_4(0) = G(G_3(0)) = G(0.9938) = 0.9982$.

This gives $P(T = 1) = G_1(0) = 0.9118$, $P(T = 2) = G_2(0) - G_1(0) = 0.9777 - 0.9118 = 0.0659$, $P(T = 3) = G_3(0) - G_2(0) = 0.9938 - 0.9777 = 0.0161$, and $P(T = 4) = G_4(0) - G_3(0) = .0044$.

d) $p/(1-m) = (1/\binom{100}{3})/(1-0.286) = 8.66147 \times 10^{-6} \approx (1.40)p$.

4.27 a) $G(s) = 1 - p + ps$.

$$G_2(s) = G(G(s)) = 1 - p + p(1 - p + ps) = 1 - p^2 + p^2 s.$$

$$G_3(s) = G_2(G(s)) = 1 - p^2 + p^2(1 - p + ps) = 1 - p^3 + p^3 s.$$

In general, $G_n(s) = 1 - p^n + p^n s$, which is the generating function of a Bernoulli distribution with parameter p^n.

b) The mean of the offspring distribution is $\mu = 0(1-p) + 1(p) = 0.9$. This is the subcritical case, and thus the extinction probability is 1. The mean of total progeny is $E(T) = 1/(1-\mu) = 1/(1-0.9) = 10$.

4.28 Branching process with immigration.

a) We have $Z_n = \left(\sum_{k=1}^{Z_{n-1}} X_k\right) + W_n$, for $n \geq 1$. By independence,

$$\begin{aligned}G_n(s) &= E\left(s^{Z_n}\right) = E\left(s^{W_n} s^{\sum_{k=1}^{Z_{n-1}} X_k}\right) = H_n(s)E\left(E\left(s^{\sum_{k=1}^{Z_{n-1}} X_k} | Z_{n-1}\right)\right)\\ &= H_n(s)E\left(E\left(s^{X_1}\right)^{Z_{n-1}}\right) = H_n(s)E\left(G(s)^{Z_{n-1}}\right) = H_n(s)G_{n-1}(G(s))\end{aligned}$$

b) Suppose the offspring distribution is Bernoulli with parameter p, and the immigration distribution is Poisson with parameter λ.

This gives $H_n(s) = e^{-\lambda(1-s)}$ and $G(s) = 1 - p + ps$. Hence

$$\begin{aligned}G_1(s) &= H_1(s)G(s) = e^{-\lambda(1-s)}(1-p+ps),\\ G_2(s) &= H_2(s)G_1(G(s)) = e^{-\lambda(1-s)}e^{-\lambda(1-(1-p+ps))}(1-p+p(1-p+ps))\\ &= e^{-\lambda(1-s)(1+p)}\left(1-p^2+p^2 s\right),\\ G_3(s) &= H_3(s)G_2(G(s)) = e^{-\lambda(1-s)}e^{-\lambda(1-(1-p+ps))(1+p)}\left(1-p^2+p^2(1-p+ps)\right)\\ &= e^{-\lambda(1-s)(1+p+p^2)}(1-p^3+p^3 s).\end{aligned}$$

The general term is

$$G_n(s) = e^{-\lambda(1-s)(1+p+\cdots+p^n)}\,(1-p^n+p^n s) = e^{-\lambda(1-s)(1-p^{n+1})/(1-p)}\,(1-p^n+p^n s)$$
$$\to e^{-\lambda(1-s)/(1-p)}, \text{ as } n \to \infty.$$

The latter is the generating function of a Poisson distribution with parameter $\lambda/(1-p)$, which is the limiting distribution of generation size.

4.29 `R`: Numerical method to find the extinction probability for the following cases.

a) $a_0 = 0.8$, $a_4 = 0.1$, $a_9 = 0.1$.

```
> pgf <- function(s) {
+  0.8 + 0.1*s^4 + 0.1*s^9  }
> x <- 0.5  # initial value
> e <- pgf(x)
> for (i in 1:100) {
+  e <- pgf(e)  }
> e
[1] 0.9152025
```

b) Offspring distribution is uniform on $\{0, 1, \ldots, 10\}$.

```
> pgf <- function(s) {
+  (1/11)*(1+s+s^2+s^3+s^4+s^5+s^6+s^7+s^8+s^9+s^10)  }
> x <- 0.5  # initial value
> e <- pgf(x)
> for (i in 1:100) {
+  e <- pgf(e)  }
> e
[1] 0.101138
```

c) $a_0 = 0.6$, $a_3 = 0.2$, $a_6 = 0.1$, $a_{12} = 0.1$.

```
> pgf <- function(s) {
+   0.6 + 0.2*s^3 + 0.1*s^6 + 0.1*s^12 }
> x <- 0.5  # initial value
> e <- pgf(x)
> for (i in 1:100) {
+  e <- pgf(e)  }
> e
[1] 0.6700263
```

4.30 `R`: Simulate branching process and extinction probability.

```
> branch <- function(n) {
+ z <- c(1,rep(0,n))
+ for (i in 2:(n+1)) {
```

```
+  z[i]<-sum( sample(0:2, z[i-1], replace=T, prob=c(1/4, 1/4, 1/2)))
+ }
+ return(z) }
> # Assume extinction occurs by 20th generation
> n <- 5000
> simlist <- replicate(n, branch(20)[21])
> sum(simlist==0)/n  # extinction probability estimate
[1] 0.5012
```

4.31 R: Simulating branching process with uniform offspring distribution.

a, b)

```
> branch <- function(n) {
+ z <- c(1,rep(0,n))
+ for (i in 2:(n+1)) {
+   z[i] <- sum( sample(0:4, z[i-1], replace=T))  }
+ return(z) }
>
> trials <- 50000
> sim1 <- numeric(trials)  # 3rd generation extinction?
> sim2 <- numeric(trials)  # long-term extinction
> for (i in 1:trials) {
+ out <- branch(10)
+ sim1[i] <- if (out[4]==0) 1 else 0  # 3rd generation
+ sim2[i] <- if (out[11] == 0) 1 else 0
+ }
> mean(sim1)  # P(Z_3 = 0)
[1] 0.26458
> mean(sim2)  # Extinction probability e
[1] 0.27394
>
> pgf <- function(s) (1/5)*(1+s+s^2+s^3+s^4)
> pgf(pgf(pgf(0)))  # G_3(0) = P(Z_3 = 0)
[1] 0.2663783
```

4.32 R: Simulate the branching process with immigration with $p = 3/4$ and $\lambda = 1.2$.

```
> branch <- function(n) {
+ z <- c(1,rep(0,n))
+ for (i in 2:(n+1)) {
+  z[i] <- sum( sample(0:1, z[i-1], replace=T,
+  prob=c(1/4, 3/4))) + rpois(1,1.2) }
+ return(z) }
> # Assume 100th generation is close to limit
> n <- 5000
> simlist <- replicate(n, branch(100)[101])
> mean(simlist)
[1] 4.791
```

```
> var(simlist)
[1] 4.871893
```

Theoretical result saws that the limit of generation size is Poisson with parameter $\lambda/(1-p) = 1.2/(1/4) = 4.8$. Simulation shows that mean and variance are close to 4.8. Could also compare the histogram of `simlist` with probability mass function of Poisson distribution.

4.33 `R`: Total progency for branching process whose offspring distribution is Poisson with parameter $\lambda = 0.60$.

```
> branch <- function(n,lam) {  ## Poisson
+   z <- c(1,rep(0,n))
+   for (i in 2:(n+1)) {
+  z[i] <- sum(rpois(z[i-1],lam))
+  }
+  return(z)   }
> # Assume extinction occurs by 50th generation
> n <- 10000
> simlist <- replicate(n, sum(branch(50,0.60)))
> mean(simlist)
[1] 2.5308
> var(simlist)
[1] 9.594811
```

4.34 `R`: Extinction function.

```
extinct <- function(x) {
e <- 0.5 # initial value
z <- length(x)-1
for (k in 1:100) e <- sum(x*e^(0:z))
return(e)
}
```

Chapter 5

5.1 Let

$$\boldsymbol{P} = \begin{matrix} & \text{Truck} & \text{Car} \\ \text{Truck} & 1/5 & 4/5 \\ \text{Car} & 1/4 & 3/4 \end{matrix},$$

with stationary distribution $\boldsymbol{\pi} = (5/21, 16/21)$. By the strong law of large numbers the toll collected is, on average,

$$1000\left(5\left(\frac{5}{21}\right) + 1.5\left(\frac{16}{21}\right)\right) = \$2333.33.$$

5.2 The random walk has stationary distribution

$$\pi_x = \begin{cases} 1/2k, & \text{if } x = 0, k \\ 1/k, & \text{if } x = 1, \ldots, k-1. \end{cases}$$

Let

$$r(x) = \begin{cases} k, & \text{if } x = 0, k \\ -1, & \text{if } x = 1, \ldots, k-1. \end{cases}$$

The long-term average reward is

$$\sum_{x=0}^{k} r(x)\pi_x = 2(k)\frac{1}{2k} - (k-1)\frac{1}{k} = \frac{1}{k}.$$

5.3 a) Compute $(10000) \times \boldsymbol{\lambda P}^2$, with $\boldsymbol{\lambda} = (0.6, 0.3, 0.1)$. This gives Car: 3645, Bus: 4165, Bike: 2190.

The stationary distribution is $\boldsymbol{\pi} = (0.2083, 0.4583, 0.3333)$. For long-term totals, compute $(10000) \times \boldsymbol{\pi}$ to get Car: 2083, Bus; 4583, Bike: 3333.

b) Current CO_2 levels: $271(0.6)+101(0.3)+21(0.1) = 195$ g; Long-term: $271(0.208) + 101(0.458) + 21(0.333) = 109.75$ g.

5.4 a) Let g_m denote the number of m-element good sequences. The number of good sequences with a 0 in the first position is g_{m-1}, as the remaining $m-1$ elements must constitute a good sequence. The number of good sequences with a 1 in the first position is g_{m-2} as the second element must be 0, and the remaining $m-2$ elements must form a good sequence. This gives $g_m = g_{m-1} + g_{m-2}$, with $g_1 = 2$ and $g_2 = 3$, which is a shifted Fibonacci sequence. That is, $g_m = f_{m+2}$.

b) Consider the $m-k$ 0s in a good sequence as an ordered list. The 0s determine $m-k+1$ positions (gaps) for the 1s of the sequence (the spaces in between the 0s as well as the leftmost and rightmost spaces. Choose k of these positions for the 1s in $\binom{m-k+1}{k}$ ways. This gives the result.

c) Use software to compute

$$\mu_m = \sum_{k=0}^{\lceil m/2 \rceil} k \frac{\binom{m-k+1}{k}}{f_{m+2}}.$$

This gives $\mu_{10} = 2.91667$; $\mu_{100} = 27.7921$; $\mu_{1000} = 276.546$.

d) $(5 - \sqrt{5})/10 = 0.276393$.

5.5 Exhibit a Metropolis-Hastings algorithm to sample from $\boldsymbol{\pi} = (0.01, 0.39, 0.11, 0.18, 0.26, 0.05)$.

```
> trans <- function(old, new) {
+  pr <- c(0.01,0.39,0.11,0.18,0.26,0.05)
+  acc <- pr[new]/pr[old]
+  if (acc >= 1) return(new) else {
+  if (runif(1) < acc) return(new) else return(old)
+  }    }
>  trials <- 20000
>  simlist <- numeric(trials)
>  for (i in 1:trials) {
+  state <- 1
+  # run the chain for 100 steps to be close to stationarity
+  for (k in 1:100) state <- trans(state,sample(1:6,1))
+  simlist[i] <- state
+  }
>  table(simlist)/trials
simlist
      1       2       3       4       5       6
0.00940 0.39185 0.11315 0.17890 0.25565 0.05105
```

5.6 Show how to generate a Poisson random variable with parameter λ using simple symmetric random walk as the proposal distribution.

Let $U \sim \text{Uniform}(0,1)$. If the walk is at state $k = 0$, then move to $k = 1$ if $U < \lambda$. Otherwise, stay at $k = 0$. If the walk is at $k \geq 1$, then flip a fair coin to let the proposal state be either $k-1$ or $k+1$. If the proposal is $k-1$, then accept the proposal (move to $k-1$) if

$$U < \frac{e^{-\lambda}\lambda^{k-1}/(k-1)!}{e^{-\lambda}\lambda^k/k!} = \frac{k}{\lambda}.$$

Otherwise, stay at k. On the other hand, if the proposal is $k+1$, then accept if

$$U < \frac{e^{-\lambda}\lambda^{k+1}/(k+1)!}{e^{-\lambda}\lambda^k/k!} = \frac{\lambda}{k+1}.$$

Otherwise, stay at k.

5.7 Exhibit a Metropolis-Hastings algorithm to sample from a binomial distribution. Use a proposal distribution which is uniform on $\{0, 1, \ldots, n\}$.

Suppose the chain is currently at state i. Let j be the proposal state, chosen uniformly on $\{0, 1, \ldots, n\}$. Let $U \sim \text{Uniform}(0,1)$. Accept j as the next state of the chain if

$$U < \frac{\binom{n}{j}p^j(1-p)^{n-j}}{\binom{n}{i}p^i(1-p)^{n-i}} = \frac{i!(n-i)!}{j!(n-j)!}\left(\frac{p}{1-p}\right)^{j-i}.$$

Otherwise, stay at state i.

5.8 a)

$$\boldsymbol{T} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

b)

$$\boldsymbol{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1/3 & 2/3 & 0 & 0 & 0 \\ 1/2 & 1/4 & 1/4 & 0 & 0 \\ 0 & 1/2 & 7/18 & 2/18 & 0 \\ 0 & 0 & 1/2 & 5/12 & 1/12 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \end{array}.$$

c) One checks that the stationary distribution is

$$\begin{aligned} \boldsymbol{\pi} &= (81/256, 27/64, 27/128, 3/64, 1/256) = (0.316, 0.422, 0.211, 0.469, 0.004) \\ &= \binom{4}{k}(1/4)^k(3/4)^{4-k}, \text{ for } k = 0, \ldots, 4. \end{aligned}$$

One also checks that the detailed balance equations $\pi_i P_{ij} = \pi_j P_{ji}$ are satisfied for all i, j.

5.9 Show how to use the Metropolis-Hastings algorithm to simulate from the *double exponential distribution*, with density

$$f(x) = \frac{\lambda}{2} e^{-\lambda|x|}, \text{ for } -\infty < x < \infty.$$

```
trials <- 10000
lambda <- 3  # choose parameter value
sim <- numeric(trials)
for (k in 1:trials) {
state <- 0
for (i in 1:50) {
# proposal drawn from a normal dist'n with mean = state, var = 1
y <- rnorm(1,state,1)
acc <- exp(-lambda*(abs(y)-abs(state)))
if (runif(1) < acc) state <- y
}
sim[k] <- state
}
## check fit
hist(sim, freq=F)
curve(lambda/2 * exp(-lambda*abs(x)),-4,4,add=T)
```

5.10 The update function is

$$g(1,u) = \begin{cases} 1, & \text{if } u < 0.5 \\ 2, & \text{if } 0.5 \le u < 1, \end{cases} \quad g(2,u) = \begin{cases} 1, & \text{if } u < 0.5 \\ 3, & \text{if } 0.5 \le u < 1, \end{cases}$$

$$\text{and} \quad g(3,u) = \begin{cases} 2, & \text{if } u < 0.5 \\ 3, & \text{if } 0.5 \le u < 1. \end{cases}$$

One checks that if $x \le y$, then $g(x,u) \le g(y,u)$, for all $u \in (0,1)$.

5.11 a) The update function is

$$g(1,u) = \begin{cases} 1, & \text{if } u < 0.5 \\ 2, & \text{if } 0.5 \le u < 1, \end{cases} \quad g(n,u) = \begin{cases} n-1, & \text{if } u < 0.5 \\ n, & \text{if } 0.5 \le u < 1, \end{cases}$$

$$\text{and for } 1 \le x \le n-1, \quad g(x,u) = \begin{cases} x-1, & \text{if } u < 0.5 \\ x+1, & \text{if } 0.5 \le u < 1. \end{cases}$$

One checks that for $1 \le x \le y \le n$, then $g(x,u) \le g(y,u)$, for all $u \in (0,1)$.

b)

```
> update <- function(state,u) {  # update function
+  z <- state
+  if (state==1 & u > 0.5) z <- 2 else {
+  if (state == n & u < 0.5) z <- n-1 else {
+  if (state > 1 & state < n & u < 0.5) z <- state - 1 else {
+  if (state > 1 & state < n) z <- state + 1 }
+  }  }
+  return(z)  }
>
> trials <- 10  # generate 10 samples
> n <- 100
> simlist <- numeric(trials)
> for (g in 1:trials) {
+ coalesce <- 0
+ s <- 1
+ u <- runif(1)
+ bottom <- 1
+ top <- n
+ while (coalesce == 0) {
+ s = s*2
+ # print(c(s,bottom,top))
+ u <- c(runif(s/2),u)
+ for (a in 1:length(u)) {
+  bottom <- update(bottom,u[a])
+  top <- update(top,u[a])
+ }
+ if (bottom==top)  coalesce <- 1
```

```
+ }
+ simlist[g] <- bottom
+ }
> simlist
 [1]  94  12  63 100  24  84  93  86  73  64
```

5.12 Consider random walk on the k-hypercube graph.

When a coordinate is selected it is replaced by the outcome of a fair coin toss, independently of any other coordinate. Thus after all k coordinates have been selected the distribution is uniform on the set of k-element binary sequences. This shows that T is a strong stationary time.

The first time that all k coordinates are selected has the same distribution as the coupon collector's problem with k coupons. The expected time to collect k coupons is $k \ln k$, for large k. This gives an upper bound on the number of steps it takes for the hypercube walk to get uniformly distributed.

5.13 Let T be the first time that all books are selected. The event $\{T > t\}$ is equal to the event that some book has not been selected by time t. The probability that book j is not selected by time n is equal to $(1 - p_j)^n$. This gives

$$\begin{aligned} v(n) \leq P(T > t) &= P\left(\bigcup_{j=1}^{k} \text{Book } j \text{ is not selected by time } n\right) \\ &\leq \sum_{j=1}^{k} (1 - p_j)^n \leq \sum_{j=1}^{k} e^{-p_j n}, \end{aligned}$$

using that $(1 - x) \leq e^{-x}$, for all x.

5.14 Let

$$\boldsymbol{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \end{array}.$$

Find the total variation distance $v(n)$.

For $n \geq$, the nth power of$\boldsymbol{P}$ is

$$\boldsymbol{P}^n = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 1/3^n & (3^n - 1)/(2 \cdot 3^n) & (3^n - 1)/(2 \cdot 3^n) \\ 0 & 1/2 & 1/2 \\ 0 & 1/2 & 1/2 \end{pmatrix} \end{array},$$

with limiting distribution $\boldsymbol{\pi} = (0, 1/2, 1/2)$.

Total variation distance is $v(n) = 3^{-n}$, for $n \geq 0$.

5.15 For the Gilbert-Shannon-Reeds model for riffle shuffles give the Markov transition matrix for a 3-card deck.

$$
\boldsymbol{P} = \begin{array}{c} \\ 123 \\ 132 \\ 213 \\ 231 \\ 312 \\ 321 \end{array} \begin{array}{c} \begin{array}{cccccc} 123 & 132 & 213 & 231 & 312 & 321 \end{array} \\ \begin{pmatrix} 1/2 & 1/8 & 1/8 & 1/8 & 1/8 & 0 \\ 1/8 & 1/2 & 1/8 & 0 & 1/8 & 1/8 \\ 1/8 & 1/8 & 1/2 & 1/8 & 0 & 1/8 \\ 1/8 & 0 & 1/8 & 1/2 & 1/8 & 1/8 \\ 1/8 & 1/8 & 0 & 1/8 & 1/2 & 1/8 \\ 0 & 1/8 & 1/8 & 1/8 & 1/8 & 1/2 \end{pmatrix} \end{array}.
$$

5.16 See the discussion on riffle shuffling and total variation distance. Bayer and Diaconis (1992) prove the following. If k cards are riffle shuffled n times with $n = (3/2)\log_2 k + \theta$, then for large k,

$$
v(n) = 1 - 2\Phi\left(\frac{-2^{-\theta}}{4\sqrt{3}}\right) + g(k),
$$

where $\Phi(x)$ is the cumulative distribution function of the standard normal distribution, and $g(k)$ is a slow-growing function of order $k^{-1/4}$.

a) Show that total variation distance tends to 1 with θ small, and to 0 with θ large.

For θ large and k large,

$$
1 - 2\Phi\left(\frac{-2^{-\theta}}{4\sqrt{3}}\right) + g(k) \approx 1 - 2\Phi(0) = 1 - 2\left(\frac{1}{2}\right) = 0.
$$

For θ small and k large,

$$
1 - 2\Phi\left(\frac{-2^{-\theta}}{4\sqrt{3}}\right) + g(k) \approx 1 - 2\Phi(-\infty) = 1 - 0 = 1.
$$

b)

$$
v(n) \approx 1 - 2\Phi\left(\frac{-2^{-(n-(3/2)\log_2 52)}}{4\sqrt{3}}\right).
$$

In R,

```
> n <- 1:10
> theta <- n - (3/2)*log(52,2)
> tv <- 1-2*pnorm(-2^(-theta)/(4*sqrt(3)))
> rbind(n,round(tv,3))
  [,1] [,2] [,3]  [,4]  [,5]  [,6]  [,7]  [,8]  [,9]  [,10]
n    1    2    3 4.000 5.000 6.000 7.000 8.000 9.000 10.000
     1    1    1 0.999 0.909 0.602 0.328 0.167 0.084  0.042
```

5.17 a, b)

$$
p(x,n) \propto \frac{e^{-3x}x^n}{n!}, \text{ for } x > 0 \text{ and } n = 0, 1, \ldots
$$

Note that

$$
p(x,n) = \frac{e^{-3x}x^n}{n!} \Bigg/ \int_0^\infty \sum_{n=0}^\infty \frac{e^{-x}x^n}{n!}\, dx = 2\frac{e^{-3x}x^n}{n!}, \text{ for } x > 0 \text{ and } n = 0, 1, \ldots
$$

The conditional distribution of N given $X = x$ is proportional to $x^n/n!$, which is a Poisson distribution with parameter x.

The conditional distribution of X given $N = n$ is proportional to $e^{-3x}x^n$, which is a gamma distribution with parameters $r = n + 1$ and $\lambda = 3$.

Having identified the conditional distributions the algorithm is implemented as follows.

```
> trials <- 100000
> simlist <- matrix(rep(0,trials*2),ncol=2) ## (X,N)
> simlist[1,] <- c(1,1)
> for (i in 2:trials) {
+   simlist[i,1] <- rgamma(1, 1 + simlist[i-1,2],3)
+   simlist[i,2] <- rpois(1,simlist[i,1])  }
> mean( (simlist[,1]^2 < simlist[,2] ))  # P(X^2 < N)
[1] 0.26458
> mean(simlist[,1]*simlist[,2])  # E(XN)
[1] 0.5066453
```

The exact value of the expectation is

$$E(XN) = \int_0^\infty \sum_{n=0}^\infty xn \frac{2e^{-3x}x^n}{n!}\,dx$$
$$= \int_0^\infty 2xe^{-3x}(e^x x)\,dx = \int_0^\infty 2x^2 e^{-2x}\,dx = \frac{1}{2}.$$

5.18 A random variable X has density function defined up to proportionality:

$$f(x) \propto e^{-(x-1)^2/2} + e^{-(x-4)^2/2}, \text{ for } 0 < x < 5.$$

```
> trials <- 1000000
> sim <- numeric(trials)
> sim[1] <- 2.45
> for (i in 2: trials) {
+  old <- sim[i-1]
+  prop <- runif(1,0,5)
+  acc <- (exp(-(prop-1)^2/2) + exp(-(prop-4)^2/2))    /
+    (   (exp(-(old-1)^2/2) + exp(-(old-4)^2/2))  )
+    if (runif(1) < acc) sim[i] <- prop else sim[i] <- old
+ }
> mean(sim)
[1] 2.498143
> var(sim)
[1] 2.099922
```

The exact mean for this symmetric distribution is $E(X) = 2.5$. The variance is $\text{Var}(X) = 2.0994\ldots$.

5.19 `R`: Use MCMC to estimate $P(10 \leq X \leq 15)$, where X has a binomial distribution with $n = 50$ and $p = 1/4$.

```
> trials <- 20000
> n <- 50
> p <- 1/4
> sim <- numeric(trials)
> for (k in 1:trials) {
+ state <- 0
+ # run chain for 60 steps to be near stationarity
+ for (i in 1:60) {
+ y <- sample(0:n,1)
+ acc <- factorial(state)*factorial(n-state)/(factorial(y)*factorial(n-y))
+  *(p/(1-p))^(y-state)
+ if (runif(1) < acc) state <- y
+ }
+ sim[k] <- if (state >= 10 & state <= 15) 1 else 0
+ }
> mean(sim)  # estimate of P(10 <= X <= 15)
[1] 0.6712
> # exact probability
> pbinom(15,n,p)-pbinom(9,n,p)
[1] 0.6732328
```

5.20 R: Consider a Poisson distribution with $\lambda = 3$ conditioned to be nonzero. Implement MCMC to simulate from this distribution, using a geometric proposal distribution with $p = 1/3$. Estimate mean and variance.

```
> trials <- 20000
> p <- 1/3
> sim <- numeric(trials)
> for (k in 1:trials) {
+ state <- 0
+ # run chain for 60 steps to be near stationarity
+ for (i in 1:60) {
+ y <- rgeom(1,p)+1
+ acc <- 3^y * factorial(state)/(3^state * factorial(y)) * (1-p)^(state-y)
+ if (runif(1) < acc) state <- y
+ }
+ sim[k] <-state
+ }
> mean(sim)
[1] 3.15885
> var(sim)
[1] 2.703452
```

Chapter 6

6.1 Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda = 1.5$. Find the following.

a)

$$\begin{aligned} P(N_1 = 2, N_4 = 6) &= P(N_1 = 2, N_4 - N_1 = 4) = P(N_1 = 2)P(N_4 - N_1 = 4) \\ &= P(N_1 = 2)P(N_3 = 4) = \left(e^{-1.5}(1.5)^2/2!\right)\left(e^{-4.5}(4.5)^4/4!\right) \\ &= 0.048. \end{aligned}$$

b)

$$P(N_4 = 6|N_1 = 2) = P(N_4 - N_1 = 4|N_1 = 2) = P(N_4 - N_1 = 4) = P(N_3 = 4) = 0.1898.$$

c)

$$P(N_1 = 2|N_4 = 6) = \frac{P(N_1 = 2, N_4 = 6)}{P(N_4 = 6)} = \frac{0.048}{e^{-6}6^6/6!} = 0.297.$$

6.2 Let $(N_t)_{t\geq 0}$ be a Poisson process with parameter $\lambda = 2$. Find the following.

a)

$$\begin{aligned} E(N_3N_4) &= E(N_3(N_4 - N_3 + N_3)) = E(N_3(N_4 - N_3)) + E(N_3^2) \\ &= E(N_3)E(N_4 - N_3) + Var(N_3) + E(N_3)^2 = E(N_3)E(N_1) + Var(N_3) + E(N_3)^2 \\ &= (6)(2) + 6 + 6^2 = 54. \end{aligned}$$

b) $E(X_3X_4) = E(X_3)E(X_4) = (1/2)^2 = 1/4$.

c)

$$\begin{aligned} E(S_3S_4) &= E(S_3(S_3 + X_4)) = E(S_3^2) + E(S_3X_4) \\ &= Var(S_3) + E(S_3)^2 + E(S_3)E(X_4) = \frac{3}{2^2} + \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)\left(\frac{1}{2}\right) = \frac{15}{4}. \end{aligned}$$

6.3 a) $P(N_{0.5} = 0) = e^{-5(0.5)} = e^{-2.5} = 0.082$.

b)

$$\begin{aligned} P(N_1 = 4, N_2 - N_1 = 6) &= P(N_1 = 4)P(N_2 - N_1 = 6) = P(N_1 = 4)P(N_1 = 6) \\ &= \left(\frac{e^{-5}5^4}{4!}\right)\left(\frac{e^{-5}5^6}{6!}\right) = 0.0257. \end{aligned}$$

c)

$$\begin{aligned} P(N_1 = 6, N_5 = 25) &= P(N_1 = 6, N_5 - N_1 = 19) = P(N_1 = 6)P(N_5 - N_1 = 19) \\ &= P(N_1 = 6)P(N_4 = 19) = \left(\frac{e^{-5}5^6}{6!}\right)\left(\frac{e^{-20}20^{19}}{19!}\right) = 0.01299. \end{aligned}$$

6.4 a)

$$P(N_{1.5} \geq 2) = 1 - P(N_{1.5} = 0) - P(N_{1.5} = 1)$$
$$= 1 - e^{-1.5} - (1.5)e^{-1.5} = 1 - (2.5)e^{-1.5} = 0.442.$$

b)

$$P(N_2 = 8, N_3 = 10) = P(N_2 = 8, N_3 - N_2 = 2) = P(N_2 = 8)P(N_1 = 2)$$
$$= \left(\frac{e^{-6}6^8}{8!}\right)\left(\frac{e^{-3}3^2}{2!}\right) = 0.023.$$

c)

$$P(N_{0.25} = 1 | N_1 = 6) = \frac{P(N_{0.25} = 1, N_1 = 6)}{P(N_1 = 6)}$$
$$= \frac{P(N_{0.25} = 1, N_1 - N_{0.25} = 5)}{P(N_1 = 6)}$$
$$= \frac{P(N_{0.25} = 1)P(N_{0.75} = 5)}{P(N_1 = 6)} = 0.356.$$

6.5 The error is that $N_3 \neq N_6 - N_3$ in the second equality. While N_3 has the same *distribution* as $N_6 - N_3$, the random variables are not equal.

6.6 Occurrences of landfalling hurricanes during an El Niño event are modeled as a Poisson process in Bove et al. (1998). The authors asserts that "During an El Niño year, the probability of two or more hurricanes making landfall in the United States is 28%." Find the rate of the Poisson process.

Let

$$0.28 = P(N_1 \geq 2) = 1 - P(N_1 = 0) - P(N_1 = 1) = 1 - e^{-\lambda} - \lambda e^{-\lambda}.$$

Numerically solving $(\lambda + 1)e^{-\lambda} = 0.72$ gives the positive solution $\lambda = 1.043$.

6.7 Ben, Max and Yolanda are at the front of three separate lines in the cafeteria waiting to be served. The serving times for the three lines follow a Poisson process with respective parameters 1, 2, and 3.

a) $P(\min(B, M, Y) = Y) = \lambda_Y / (\lambda_B + \lambda_M + \lambda_Y) = 3/(1 + 2 + 3) = 1/2$.

b)

$$P(B < Y) = \int_0^\infty P(B < y | Y = y) 3e^{-3y}\, dy = \int_0^\infty P(B < y) 3e^{-3y}\, dy$$
$$= \int_0^\infty (1 - e^{-y}) 3e^{-3y}\, dy = \frac{1}{4}.$$

c) The waiting time of the first person served has an exponential distribution with parameter $\lambda_B + \lambda_M + \lambda_Y = 1 + 2 + 3 = 6$. The expected waiting time is 1/6.

6.8 Starting at 6 am, cars, buses, and motorcycles arrive at a highway toll booth according to independent Poisson processes. Cars arrive about once every 5 minutes. Buses arrive about once every 10 minutes. Motorcycles arrive about once every 30 minutes.

a)

$$P(C_{20} = 2, B_{20} = 0, M_{20} = 1) = P(C_{20} = 2)P(B_{20} = 0)P(M_{20} = 1).$$

```
> dpois(2,20/5)*dpois(0,20/10)*dpois(1,20/30)
[1] 0.00678738
```

b) The arrival process of vehicles is a Poisson process with parameter $1/5 + 1/10 + 1/30 = 1/3$ The thinned process of cars with exact change is a Poisson process with parameter $(1/4)(1/3) = 1/12$. The desired probability is $e^{-10/12} = 0.435$.

c)

$$\begin{aligned} P(S_7 - S_3 < 45) &= P(S_3 + X_4 + X_5 + X_6 + X_7 - S_3 < 45) \\ &= P(X_4 + X_5 + X_6 + X_7 < 45) = P(Z < 45), \end{aligned}$$

where Z has a gamma distribution with parameters 4 and 1/30.

```
> pgamma(45,4, 1/30)
[1] 0.06564245
```

d) Let Z_t be the number of buses and motorcycles which arrive in an interval of length t. The process of bus and motorcycle arrival is a Poisson process with parameter $1/10 + 1/30 = 2/15$. The length of the interval T between the 3rd and 4th car arrival is exponentially distributed with parameter 1/5. Conditioning on the length of the interval gives

$$\begin{aligned} P(Z_T > 0) &= \int_0^\infty P(Z_T > 0|T = t)(1/5)e^{-t/5}\,dt \\ &= \frac{1}{5}\int_0^\infty \left(1 - e^{-2t/15}\right) e^{-t/5}\,dt \\ &= 1 - \frac{1}{5}\int_0^\infty e^{-t/3}\,dt \\ &= 1 - \frac{3}{5}\int_0^\infty (1/3)e^{-t/3}\,dt = 1 - \frac{3}{5} = \frac{2}{5}. \end{aligned}$$

6.9 Let X be geometrically distributed with parameter p. The cumulative distribution function for X is

$$P(X \le x) = \sum_{k=1}^{x} P(X = k) = \sum_{k=1}^{x}(1-p)^{k-1}p = p\frac{1-(1-p)^x}{1-(1-p)} = 1 - (1-p)^x.$$

This gives,

$$P(X > s + t|X > s) = \frac{P(X > s + t)}{P(X > s)} = \frac{(1-p)^{s+t}}{(1-p)^s} = (1-p)^t = P(X > t).$$

6.10 Shown $S_n = X_1 + \cdots + X_n$ has a gamma distribution with parameters n and λ.

a) If X has a gamma distribution with parameters n and λ, then

$$m_X(t) = E\left(e^{tX}\right) = \int_0^\infty e^{tx} \frac{t^{n-1}\lambda^n e^{-\lambda x}}{\Gamma(n)}\, dx = \left(\frac{\lambda}{\lambda - t}\right)^n.$$

The moment-generating function of S_n is

$$\begin{aligned} m_{S_n}(t) &= E\left(e^{tS_n}\right) = E\left(e^{t(X_1+\cdots+X_n)}\right) \\ &= E\left(\prod_{i=1}^n e^{tX_i}\right) = \prod_{i=1}^n E\left(e^{tX_i}\right) = \left[E\left(e^{tX_1}\right)\right]^n = \left(\frac{\lambda}{\lambda - t}\right)^n. \end{aligned}$$

Thus, S_n has a gamma distribution with parameters n and λ.

b) We show the result by induction on n. The result is true for $n = 1$, as the gamma distribution in that case reduces to the exponential distribution with a parameter λ. Assume result true for all $k < n$. Observe that $S_n = X_1 + \cdots + X_{n-1} + X_n = S_{n-1} + X_n$. Further, S_{n-1} and X_n are independent, since the X_i are independent. This gives

$$P(S_n \le x) = P(S_{n-1} + X_n \le x) = \int_{-\infty}^\infty P(X_n \le x - y) f_{S_{n-1}}(y)\, dy.$$

Differentiating with respect to x,

$$\begin{aligned} f_{S_n}(x) &= \int_{-\infty}^\infty f_{X_n}(x-y) f_{S_{n-1}}(y)\, dy \\ &= \int_0^x \lambda e^{-\lambda(x-y)} \frac{\lambda e^{-\lambda y}(\lambda y)^{n-2}}{(n-2)!}\, dy \\ &= e^{-\lambda x} \frac{\lambda^n}{(n-2)!} \int_0^x y^{n-2}\, dy = e^{-\lambda x} \frac{\lambda^n}{(n-2)!} \frac{x^{n-1}}{n-1} \\ &= \lambda e^{-\lambda x} \frac{(\lambda x)^{n-1}}{(n-1)!}, \end{aligned}$$

which gives the result.

6.11 Show that a continuous probability distribution which is memoryless must be exponential.

Let X be a memoryless, continuous random variable. Let $g(t) = P(X > t)$. By memorylessness,

$$P(X > t) = P(X > s + t | X > s) = \frac{P(X > s + t)}{P(X > s)}.$$

Thus, $g(s + t) = g(s)g(t)$. It follows that

$$g(t_1 + \cdots + t_n) = g(t_1) \cdots g(t_n).$$

Let $r = p/q = \sum_{i=1}^p (1/q)$. Then $g(r) = (g(1/q))^p$. Also,

$$g(1) = g\left(\sum_{i=1}^q \frac{1}{q}\right) = g\left(\frac{1}{q}\right)^q,$$

or $g(1/q) = g(1)^{1/q}$. This gives

$$g(r) = g(1)^{p/q} = g(1)^r = e^{r \ln g(1)},$$

for all rational r. By continuity, for all $t > 0$, $g(t) = e^{-\lambda t}$, where

$$\lambda = -\ln g(1) = -\ln P(X > 1).$$

6.12 See Example 6.10. The number of diners in the restaurant at 2 p.m. has a Poisson distribution with mean and variance $5p_{120}$, where

$$p_t = \int_0^t e^{-x/40}\,dx = 40\left(1 - e^{-t/40}\right),$$

with $p_{120} = 40(1 - e^{-3}) = 38.0085$. There are approximately $5(38) = 190$ customers in the restaurant at 2 p.m., give or take about $\sqrt{190} \approx 14$ customers.

6.13 a) $s < t$

For $k = 0, 1, \ldots, n$,

$$\begin{aligned}
P(N_s = k | N_t = n) &= \frac{P(N_s = k, N_t = n)}{P(N_t = n)} = \frac{P(N_s = k, N_t - N_s = n - k)}{P(N_t = n)} \\
&= \frac{P(N_s = k)P(N_{t-s} = n - k)}{P(N_t = n)} \\
&= \frac{e^{-\lambda s}(\lambda s)^k/k! e^{-\lambda(t-s)}(\lambda(t-s))^{n-k}/(n-k)!}{e^{-\lambda t}(\lambda t)^n/n!} \\
&= \frac{n!}{k!(n-k)!}\left(\frac{s}{t}\right)^k \left(1 - \frac{s}{t}\right)^{n-k}.
\end{aligned}$$

The conditional distribution of N_s given $N_t = n$ is binomial with parameters n and $p = s/t$.

b) $s > t$

For $k \geq n$,

$$\begin{aligned}
P(N_s = k | N_t = n) &= P(N_s - N_t = k - n | N_t = n) = P(N_s - N_t = k - n) \\
&= P(N_{s-t} = k - n) = e^{-\lambda(s-t)}(s-t)^{k-n}/(k-n)!.
\end{aligned}$$

The conditional distribution is a shifted Poisson distribution with parameter $\lambda(s - t)$.

6.14 Let N_t denote the arrival process for blue cars. The event that k cars arrive between two successive red cars, given that the time between these cars is t, is equal to the event that $N_t = k$. Conditioning on the length of the interval,

$$\begin{aligned} P(X = k) &= \int_0^\infty P(N_t = k) re^{-rt}\, dt \\ &= \int_0^\infty \frac{e^{-bt}(bt)^k}{k!} re^{-rt}\, dt \\ &= \frac{b^k r}{k!} \int_0^\infty e^{-(b+r)t} t^k\, dt \\ &= \frac{b^k r}{(b+r)^{k+1}} \\ &= \frac{r}{b+r}\left(\frac{b}{b+r}\right)^k, \text{ for } k = 0, 1, \ldots, \end{aligned}$$

where the penultimate equality uses that the previous integrand is proportional to a gamma density.

6.15 The failure process is Poisson with parameter $1.5 + 3.0 = 4.5$. Let M_t and m_t denote the thinned major and minor processes, respectively.

a) $P(N_1 = 2) = e^{-4.5}(4.5)^2/2! = 0.112$.

b) $e^{-1.5/2} = 0.472$.

c)

$$\begin{aligned} P(M_2 \geq 2 \text{ or } m_2 \geq 2) &= 1 - P(M_2 \leq 1, m_2 \leq 1) = 1 - P(M_2 \leq 1)P(m_2 \leq 1) \\ &= 1 - (e^{-3} + 3e^{-3})(e^{-6} + 6e^{-6}) = 1 - 28e^{-9} = 0.997. \end{aligned}$$

6.16 a)

$$\frac{e^{-2(3/4)}(2(3/4))^3}{3!} = 0.1255.$$

b) $1 - e^{-2/7} = 0.2485$.

c) Let A_t denote the process of alcohol-related accidents. Let S_t denote the process of non-alcohol-related accidents. The desired probability is

$$\begin{aligned} P(A_4 < 3 | N_4 = 6) &= \frac{P(A_4 < 3, N_4 = 6)}{P(N_4 = 6)} = \frac{\sum_{k=0}^2 P(A_4 = k, S_4 = 6 - k)}{P(N_4 = 6)} \\ &= \frac{1}{e^{-8}8^6/6!} \sum_{k=0}^2 P(A_4 = k) P(S_4 = 6 - k) \\ &= \frac{1}{e^{-8}8^6/6!} \sum_{k=0}^2 e^{-6}6^k/k! e^{-2}2^{6-k}/(6-k)! \\ &= \sum_{k=0}^2 \binom{6}{k} \left(\frac{3}{4}\right)^k \left(\frac{1}{4}\right)^{6-k} = 0.0376. \end{aligned}$$

6.17

$$\begin{aligned}
E\left(\sum_{n=1}^{N_t} S_n^2\right) &= \sum_{k=0}^{\infty} E\left(\sum_{n=1}^{N_t} S_n^2 | N_t = k\right) \frac{e^{-\lambda t}(\lambda t)^k}{k!} \\
&= \sum_{k=0}^{\infty} E\left(\sum_{n=1}^{k} U_{(n)}^2\right) \frac{e^{-\lambda t}(\lambda t)^k}{k!} = \sum_{k=0}^{\infty} E\left(\sum_{n=1}^{k} U_n^2\right) \frac{e^{-\lambda t}(\lambda t)^k}{k!} \\
&= \sum_{k=0}^{\infty} \sum_{n=1}^{k} E\left(U_n^2\right) \frac{e^{-\lambda t}(\lambda t)^k}{k!} = \sum_{k=0}^{\infty} \frac{kt^2}{3} \frac{e^{-\lambda t}(\lambda t)^k}{k!} = \frac{\lambda t^3}{3}.
\end{aligned}$$

6.18 a) The event $\{X > t\}$ is equal to the event that there are no planets in the sphere, centered at the Death Star, of radius t. This gives $P(X > t) = e^{-4\pi t^3/3}$, for $t > 0$. Taking complements and differentiating gives the density

$$f_X(t) = 4\pi t^2 e^{-4\pi t^3/3}, \text{ for } t > 0.$$

b)

$$E(X) = \int_0^{\infty} t\left(4\pi t^2 e^{-4\pi t^3/3}\right) dt = 0.55396.$$

The integral is solved by numerical methods.

6.19 See the section on embedding. In the notation of the book $T = \min_{1 \le k \le 365} Z_k$, where the Z_k are i.i.d. gamma random variables with parameters $n = 3$ and $\lambda = 1/365$. This gives

$$\begin{aligned}
E(T) &= \int_0^{\infty} P(T > t)\, dt = \int_0^{\infty} P(\min(Z_1, \ldots, Z_{365}) > t)\, dt \\
&= \int_0^{\infty} P(Z_1 > t)^{365}\, dt = \int_0^{\infty} \left(\int_t^{\infty} \frac{x^2 e^{-x/365}}{365^3 \times 2}\, dx\right)^{365} dt \\
&= \frac{1}{266450} \int_0^{\infty} e^{-t/365}(266450 + 730t + t^2)\, dt = 88.74.
\end{aligned}$$

The integral is solved by numerical integration.

6.20

$$\begin{aligned}
P(N_B = 1 | N_A = 1) &= \frac{P(N_B = 1, N_A = 1)}{P(N_A = 1)} = \frac{P(N_B = 1, N_{AB^c} = 0)}{P(N_A = 1)} \\
&= \frac{P(N_B = 1)P(N_{AB^c} = 0)}{P(N_A = 1)} = \frac{e^{-\lambda|B|}\lambda|B|e^{-\lambda(|A|-|B|)}}{e^{-\lambda|A|}\lambda|A|} \\
&= \frac{|B|}{|A|}.
\end{aligned}$$

6.21 For $s < t$,

$$\begin{aligned}
\text{Corr}(N_s, N_t) &= \frac{\text{Cov}(N_s, N_t)}{SD(N_s)SD(N_t)} = \frac{E(N_s N_t) - E(N_s)E(N_t)}{SD(N_s)SD(N_t)} \\
&= \frac{E(N_s N_t) - \lambda^2 st}{\sqrt{\lambda s}\sqrt{\lambda t}} = \frac{E(N_s N_t) - \lambda^2 st}{\lambda\sqrt{st}}.
\end{aligned}$$

Also,

$$E(N_sN_t) = E(N_s(N_t - N_s + N_s)) = E(N_s(N_t - N_s)) + E(N_s^2)$$
$$= E(N_s)E(N_t - N_s) + (\lambda s + \lambda^2 s^2) = (\lambda s)(\lambda(t-s)) + \lambda s + \lambda^2 s^2 = \lambda^2 st + \lambda s.$$

This gives

$$\text{Corr}(N_s, N_t) = \frac{E(N_sN_t) - \lambda^2 st}{\lambda\sqrt{st}} = \frac{\lambda s}{\lambda\sqrt{st}} = \sqrt{\frac{s}{t}}.$$

6.22 Poisson Casino.

Conditioning on the number of rolls in $[0, t]$, the desired probability is

$$\sum_{n=0}^{\infty} \left(\frac{1}{6}\right)^n e^{-\lambda t} \frac{(\lambda t)^n}{n!} = e^{-5\lambda t/6}.$$

6.23 a) Let X denote the given region. Then

$$P(O_X \geq 1, M_X \geq 1) = P(O_X \geq 1)P(M_X \geq 1) = (1 - e^{-\lambda_O x})(1 - e^{-\lambda_M x}).$$

b) Let R denote the given region. The area of R is $\pi 120^2 - \pi 100^2 = 4400\pi$. The desired probability is

$$P(O_R \geq 1, M_R = 0) = P(O_R \geq 1)P(M_R = 0) = (1 - e^{-\lambda_O 4400\pi})e^{-\lambda_M 4400\pi}.$$

6.24 a) $P(M < S) = \lambda/(\lambda + \mu)$.

b) Let $N_t = M_t + S_t$ be the superposition process. Then N_t is a Poisson process with parameter $\lambda + \mu$. The desired probability is $P(N_t = 1) = (\lambda + \mu)e^{-(\lambda+\mu)}$.

c) $P(S_2 = 1, M_2 = 2) = P(S_2 = 1)P(M_2 = 2) = e^{-2\mu}(2\mu)e^{-2\lambda}(2\lambda)^2/2 = 4\lambda^2\mu e^{-2(\lambda+\mu)}$.

6.25 Given that 10 computers failed, the time M of the last failure has the same distribution as the maximum of 10 uniform random variables distributed on $(0, 7)$ (units are days). Such a maximum has density function

$$f_M(t) = \frac{10t^9}{7^{10}}, \text{ for } 0 < t < 7,$$

with mean

$$E(M) = \int_0^7 t\frac{10t^9}{7^{10}}\, dt = \frac{70}{11} = 6.36.$$

The expected time of failure was 8:43 am on the last day of the week.

6.26

$$G(s) = E\left(s^{N_t}\right) = \sum_{k=0}^{\infty} s^k \frac{e^{-\lambda t}(\lambda t)^k}{k!} = e^{-\lambda t}e^{\lambda ts} = e^{-\lambda t(1-s)}.$$

6.27 The votes for each party represent independent thinned processes. Let $N_t = A_t + B_t + C_t + D_t + E_t + F_t$.

a) We have that $A_t + B_t + C_t + D_t$ is a Poisson process with parameter $60t(0.05 + 0.30 + 0.10 + 0.10) = 33t$. The desired probability is

$$\begin{aligned} P(N_2 > 100 | E_2 + F_2 = 40) &= P(A_2 + B_2 + C_2 + D_2 > 60) \\ &= \sum_{k=61}^{\infty} \frac{e^{-2(33)}(2(33)^k}{k!} = 0.747. \end{aligned}$$

b) $P(N_{1/60} \geq 1) = 1 - e^{-1} = 0.632$.

c) $\lambda_C/(\lambda_B + \lambda_C + \lambda_D) = (0.10)/(0.30 + 0.10 + 0.10) = 0.20$.

6.28 Let $X_1, X_2, \ldots$ be an i.i.d. sequence where X_i is the number of claims filed after the ith tornado. Let T denote the total number of claims filed. Then $T = \sum_{i=1}^{N_t} X_i$. Using results for random sums of random variables, as in Examples 1.28 and 1.33,

$$E(T) = E(N_t)E(X_1) = (2t)(30) = 60t,$$

and

$$Var(T) = Var(X_1)E(N_t) + [E(X_1)]^2 Var(N_t) = (30)(2t) + 30^2(2t) = 1860t,$$

with $SD(T) = \sqrt{1860t}$.

6.29 Let p be the probability that a job offer is acceptable. Then $p = e^{-35/25} = 0.246$. The arrival of acceptable job offers is a thinned Poisson process with rate $\lambda = 2p$. Let T be the time of the first acceptable job offer. Then T has an exponential distribution with parameter $2p$. The desired probability is $P(T < 3) = 0.77$.

6.30 In Example 6.9 it is shown that $E\left(\sum_{k=1}^{N_t}(t - S_k)|N_t = n\right) = tn/2$, and thus

$$E\left(\sum_{k=1}^{N_t}(t - S_k)|N_t\right) = \frac{tN_t}{2}.$$

Also,

$$\begin{aligned} Var\left(\sum_{k=1}^{N_t}(t - S_k)|N_t = n\right) &= Var\left(\sum_{k=1}^{n}(t - S_k)|N_t = n\right) \\ &= Var\left(\sum_{k=1}^{n} S_k|N_t = n\right) = Var\left(\sum_{k=1}^{n} U_{(k)}\right) = Var\left(\sum_{k=1}^{n} U_k\right), \end{aligned}$$

where $U_1, \ldots, U_n$ are i.i.d. random variables uniformly distributed on $[0, t]$. This gives

$$Var\left(\sum_{k=1}^{N_t}(t - S_k)|N_t = n\right) = Var\left(\sum_{k=1}^{n} U_k\right) = \sum_{k=1}^{n} Var(U_k) = \frac{t^2 n}{12},$$

and thus $Var\left(\sum_{k=1}^{N_t}(t-S_k)|N_t\right) = t^2N_t/12$. Finally, by the law of total variance,

$$\begin{aligned}
Var\left(\sum_{k=1}^{N_t}(t-S_k)\right) &= Var(E\left(\sum_{k=1}^{N_t}(t-S_k)|N_t\right)) + E\left(Var\left(\sum_{k=1}^{N_t}(t-S_k)|N_t\right)\right)\\
&= Var\left(\frac{tN_t}{2}\right) + E\left(\frac{t^2N_t}{12}\right)\\
&= \frac{t^2(\lambda t)}{4} + \frac{t^2(\lambda t)}{12} = \frac{\lambda t^3}{3}.
\end{aligned}$$

6.31 The expected average waiting time is $E\left(\frac{1}{N_t}\sum_{k=1}^{N_t}(t-S_k)\right)$. Near duplicating the solution in Example 6.9, the desired expectation is $t/2$.

6.32

$$\begin{aligned}
E\left(\sum_{i=1}^{N_t} e^{-rS_i}\right) &= \sum_{n=0}^{\infty} E\left(\sum_{i=1}^{N_t} e^{-rS_i}|N_t=n\right)P(N_t=n)\\
&= \sum_{n=0}^{\infty} E\left(\sum_{i=1}^{n} e^{-rU_i}\right)P(N=n)\\
&= \sum_{n=0}^{\infty} n\left(\int_0^t \frac{e^{-rs}}{t}\,ds\right)P(N=n)\\
&= \frac{1-e^{-rt}}{tr}\sum_{n=0}^{\infty} nP(N_t=n)\\
&= \frac{\lambda(1-e^{-rt})}{r}.
\end{aligned}$$

6.33 As $S_n = X_1 + \cdots + X_n$, and the X_i are i.i.d. random variables,

$$S_n = E(S_n|S_n) = E(X_1+\cdots+X_n|S_n) = \sum_{i=1}^{n} E(X_i|S_n) = \sum_{i=1}^{n} E(X_1|S_n) = nE(X_1|S_n).$$

This gives $E(S_1|S_n) = E(X_1|S_n) = S_n/n$.

6.34 S_{N_t} is the time of the last arrival in $[0,t]$, or 0, if there are no arrivals. The distribution is obtained by conditioning on N_t. If $N_t = 0$, then $S_{N_t} = S_0 = 0$. This occurs with probability $P(N_t = 0) = e^{-\lambda t}$. Otherwise, for $0 < x \le t$,

$$\begin{aligned}
P(S_{N_t} \le x) &= \sum_{n=0}^{\infty} P(S_{N_t} \le x|N_t=n)P(N_t=n)\\
&= \sum_{n=0}^{\infty} P(S_n \le x|N_t=n)\frac{e^{-\lambda t}(\lambda t)^n}{n!}\\
&= \sum_{n=0}^{\infty} P(M \le x)\frac{e^{-\lambda t}(\lambda t)^n}{n!},
\end{aligned}$$

where M is the maximum of n i.i.d. random variables uniformly distributed on $[0, t]$. This gives

$$P(S_{N_t} \leq x) = \sum_{n=0}^{\infty} \left(\frac{x}{t}\right)^n \frac{e^{-\lambda t}(\lambda t)^n}{n!} = e^{-\lambda t} \sum_{n=0}^{\infty} \frac{(\lambda x)^n}{n!} = e^{-\lambda t} e^{\lambda x} = e^{-\lambda(t-x)}.$$

Differentiating with respect to x,

$$f_{S_{N_t}}(x) = \lambda e^{-\lambda(t-x)}, \text{ for } 0 \leq x \leq t.$$

The distribution of S_{N_t} is mixed. It has discrete and continuous components. The cumulative distribution function is

$$P(S_{N_t} \leq x) = \begin{cases} 0, & \text{if } x < 0 \\ e^{-\lambda(t-x)}, & \text{if } 0 \leq x < t \\ 1, & \text{if } x \geq t. \end{cases}$$

6.35 The desired probability is

$$P(N_C = 0) = e^{-\iint_C \lambda(x,y)\, dx\, dy}.$$

With polar coordinates,

$$\begin{aligned} \iint_C \lambda(x,y)\, dx\, dy &= \iint_C e^{-(x^2+y^2)}\, dx\, dy \\ &= \int_0^{2\pi} \int_0^1 e^{-r^2} r\, dr\, d\theta \\ &= 2\pi \int_0^1 \frac{1}{2} e^{-u}\, du = (1 - e^{-1})\pi. \end{aligned}$$

This gives

$$P(N_C = 0) = e^{(1-e^{-1})\pi} = 0.137.$$

6.36 Let $(N_t)_{t \geq 0}$ be the non-homogenous Poisson process of customer arrivals. For $k \geq 0$,

$$P(N_3 = k) = \frac{e^{-\int_0^3 t^2\, dt} \left(\int_0^3 t^2\, dt\right)^k}{k!} = \frac{e^{-9} 9^k}{k!}.$$

6.37 The process of serious injuries is a compound Poisson process, which is a random sum of random variables. For results for such sums see Examples 1.28 and 1.33. We have that

$$E(C_t) = E(N_t)E(X_1) = (3)(52)(2) = 312 \text{ serious injuries,}$$

and

$$Var(C_t) = Var(X_1)E(N_t) + E(X_1)^2 Var(N_t) = (2)(52)(3) + (2^2)(52)(3) = 936,$$

with standard deviation $SD(C_t) = \sqrt{936} = 30.594$ serious injuries.

6.38

$$P(N_t = n) = \int_0^\infty P(N_t = n | \Lambda = \lambda) \mu e^{-\mu\lambda} \, d\lambda = \int_0^\infty \frac{e^{-\lambda t}(\lambda t)^n}{n!} \mu e^{-\mu\lambda} \, d\lambda$$
$$= \left(\frac{\mu}{\mu + t}\right)\left(\frac{t}{\mu + t}\right)^n, \text{ for } n = 0, 1, \ldots.$$

6.39

$$P(N_1 = 1) = \int_0^1 P(N_1 = 1 | \Lambda = \lambda) \, d\lambda = \int_0^1 \lambda e^{-\lambda} \, d\lambda = \frac{e - 2}{e} = 0.264.$$

6.40

$$P(N_t = k) = \int_0^\infty P(N_t = k | \Lambda = x) \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \, dx$$
$$= \int_0^\infty \frac{e^{-xt}(xt)^k}{k!} \frac{\lambda^n x^{n-1} e^{-\lambda x}}{(n-1)!} \, dx$$
$$= \frac{\lambda^n t^k}{k!(n-1)!} \int_0^\infty x^{n+k-1} e^{-x(\lambda+t)} \, dx$$
$$= \frac{\lambda^n t^k}{k!(n-1)!} \frac{(k+n-1)!}{(\lambda+t)^{k+n}}$$
$$= \binom{n+k-1}{k} \left(\frac{\lambda}{\lambda+t}\right)^n \left(\frac{t}{\lambda+t}\right)^k, \text{ for } k = 0, 1, \ldots.$$

6.41 `R`: The goal scoring Poisson process has parameter $\lambda = 2.68/90$. Each team's goals can be considered the outcomes of two independent thinned processes, each with parameter $\lambda/2$. By conditioning on the number of goals scored in a 90-minute match, the desired probability is

$$\sum_{k=0}^\infty \left(\frac{e^{-90\lambda/2}(90\lambda/2)^k}{k!}\right)^2 = \sum_{k=0}^\infty \left(\frac{e^{-1.34} 1.34^k}{k!}\right)^2 = 0.259.$$

```
> trials <- 100000
> mean(rpois(trials,1.34)==rpois(trials,1.34))
[1] 0.25952
```

6.42 `R`: Simulate the restaurant results.

```
> trials <- 10000
> sim <- numeric(trials)
> for (i in 1:trials) {
+ cust <- rpois(1,5*60*2)  # no. of customers in 2 hours
+ arr <- sort(runif(cust,0,120))
+ times <- arr + rexp(cust,1/40)
+ sim[i] <- sum (times > 120)   # no of customers at 2 pm
+ }
```

```
> mean(sim)
[1] 190.1673
> var(sim)
[1] 191.0688
```

6.43 `R`: Simulate a spatial Poisson process with $\lambda = 10$ on the box of volume 8 with vertices at $(\pm 1, \pm 1, \pm 1)$. Estimate the mean and variance of the number of points in the ball centered at the origin of radius 1. Compare to the exact values.

The number of points in the ball has a Poisson distribution with mean and variance $\lambda = 80(4/3)\pi/8 = 41.88\ldots$.

```
> trials <- 10000
> sim <- numeric(trials)
> for (k in 1:trials) {
+ npoints <- rpois(1,10*8)
+ points <- matrix(runif(npoints*3,-1,1),ncol=3)
+ g <- 0
+ for (i in 1:npoints) {
+   g <- g + if ( (sum(points[i,]^2) ) < 1) 1 else 0   }
+ sim[k] <- g}
> mean(sim)
[1] 41.9009
> var(sim)
[1] 42.70095
```

6.44 `R`: Parameter values are $\lambda = 50$, $t = 10$, and $r = 0.04$. Exact total present value is

$$1000\lambda \frac{1 - e^{-rt}}{r} = 1000(50)\frac{1 - e^{-0.4}}{0.04} = 412{,}100.$$

```
 > trials <- 10000
> sim <- numeric(trials)
> for (k in 1:trials) {
+  nbonds <- rpois(1,50*10)
+  times <- runif(nbonds,0,10)
+  totalpv <- sum(1000*50*(1-exp(-0.04*10))/0.04)
+  sim[k] <- totalpv
+  }
>  mean(sim)
[1] 412099.9
```

6.45 `R`: Birthday problem.

```
> sim <- numeric(trials)
> for (i in 1:trials) {
+  flag <- 0
+  ct <- 0
+  vect <- numeric(365)
```

```
+  while (flag==0) {
+  newp <- sample(1:365,1)
+  vect[newp] <- vect[newp]+1
+  if (sum(vect==3)==1) flag <- 1
+  ct <- ct + 1
+  }
+  sim[i] <- ct
+  }
>  mean(sim)
[1] 88.4412
```

Chapter 7

7.1

$$\boldsymbol{Q} = \begin{array}{c} \\ a \\ b \\ c \end{array} \begin{array}{c} \begin{array}{ccc} a & b & c \end{array} \\ \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 2 & 2 & -4 \end{pmatrix} \end{array} \qquad \widetilde{\boldsymbol{P}} = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{pmatrix}$$

$q_a = 1, q_b = 2, q_c = 4$.

7.2 a)

$$\boldsymbol{Q} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} -3 & 1 & 0 & 2 \\ 0 & -1 & 1 & 0 \\ 1 & 2 & -5 & 2 \\ 1 & 0 & 2 & -3 \end{pmatrix} \end{array} \qquad \widetilde{\boldsymbol{P}} = \begin{pmatrix} 0 & 1/3 & 0 & 2/3 \\ 0 & 0 & 1 & 0 \\ 1/5 & 2/5 & 0 & 2/5 \\ 1/3 & 0 & 2/3 & 0 \end{pmatrix}$$

$q_1 = 3, q_2 = 1, q_3 = 5, q_4 = 3$.

b) $1/q_1 = 1/3$ of an hour; c) $1/q_{34} = 1/2$ of an hour; d) The stationary distribution of the embedded Markov chain is $\boldsymbol{\pi} = (0.161, 0.204, 0.376, 0.258)$. The desired long-term proportion is 20.4%.

7.3

$$\boldsymbol{Q} = \begin{pmatrix} -a & a/2 & a/2 \\ b/2 & -b & b/2 \\ c/2 & c/2 & -c \end{pmatrix}.$$

$$\boldsymbol{\pi} = \left(\frac{bc}{ac+bc+ab}, \frac{ac}{ac+bc+ab}, \frac{ab}{ac+bc+ab} \right).$$

7.4

$$\boldsymbol{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ \vdots \end{array} \begin{array}{c} \begin{array}{cccccccc} 0 & 1 & 2 & 3 & 4 & 5 & 6 & \cdots \end{array} \\ \begin{pmatrix} -2 & 2 & 0 & 0 & 0 & 0 & 0 & \cdots \\ 1 & -3 & 2 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 2 & -4 & 2 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 3 & -5 & 2 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 3 & -5 & 2 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 3 & -5 & 2 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 3 & -5 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \cdots \end{pmatrix} \end{array}.$$

7.5

$$\boldsymbol{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -2 & 2 & 0 & 0 \\ 1 & -3 & 2 & 0 \\ 0 & 2 & -4 & 2 \\ 0 & 0 & 3 & -3 \end{pmatrix} \end{array}.$$

7.6
```
> library(expm)
> Q <- matrix(c(-2,1,1,0,1,-3,1,1,2,2,-4,0,1,2,3,-6),nrow=4,byrow=T)
> colnames(Q) <- 1:4
> rownames(Q) <- 1:4
> Q
   1  2  3  4
1 -2  1  1  0
2  1 -3  1  1
3  2  2 -4  0
4  1  2  3 -6
> P <- function(t) expm(t*Q)
> P(1)
          1         2         3          4
1 0.4295490 0.3058358 0.2159691 0.04864609
2 0.3841837 0.3331484 0.2241560 0.05851194
3 0.4022364 0.3197951 0.2252290 0.05273953
4 0.3923706 0.3238885 0.2275139 0.05622700
> P(1)[1,3]
[1] 0.2159691
> P(100)
          1         2         3          4
1 0.4070796 0.3185841 0.2212389 0.05309735
2 0.4070796 0.3185841 0.2212389 0.05309735
3 0.4070796 0.3185841 0.2212389 0.05309735
4 0.4070796 0.3185841 0.2212389 0.05309735
> P(3)[4,1] * P(1)[3,4]
[1] 0.0214681
```

7.7 a)

$$
\begin{aligned}
P'_{11}(t) &= -P_{11}(t) + 3P_{13}(t)\\
P'_{12}(t) &= -2P_{12}(t) + P_{11}(t)\\
P'_{13}(t) &= -3P_{13}(t) + 2P_{12}(t)\\
P'_{21}(t) &= -P_{21}(t) + 3P_{23}(t)\\
P'_{22}(t) &= -2P_{22}(t) + P_{21}(t)\\
P'_{23}(t) &= -3P_{23}(t) + 2P_{22}(t)\\
P'_{31}(t) &= -P_{31}(t) + 3P_{33}(t)\\
P'_{32}(t) &= -2P_{32}(t) + P_{31}(t)\\
P'_{33}(t) &= -3P_{33}(t) + 2P_{32}(t)
\end{aligned}
$$

b)

$$\boldsymbol{P}(t) = \begin{pmatrix} -1 & 0 & 1 \\ -3 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-4t} & 0 & 0 \\ 0 & e^{-2t} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1/4 & 0 & 1/4 \\ -3/2 & 1 & 1/2 \\ 3/4 & 0 & 1/4 \end{pmatrix}$$
$$= \frac{1}{4} \begin{pmatrix} 3 + e^{-4t} & 0 & 1 - e^{-4t} \\ 3 + 3e^{-4t} - 6e^{-2t} & 4e^{-2t} & 1 - 3e^{-4t} + 2e^{-2t} \\ 3 - 3e^{-4t} & 0 & 1 + 3e^{-4t}. \end{pmatrix}$$

7.8 Consider the Jukes-Cantor model for DNA nucleotide substitution.

Labeling the states a, c, g, t, consider $P_{ac}(t)$. By the backward equation,

$$\begin{aligned} P'_{ac}(t) &= -P_{ac}(t)q_c + P_{aa}(t)q_{ac} + P_{ag}(t)q_{gc} + P_{at}(t)q_{tc} \\ &= -3rP_{ac}(t) + rP_{aa}(t) + rP_{ag}(t) + rP_{at}(t) \\ &= -3rP_{ac}(t) + r(1 - P_{ac}(t) - P_{ag}(t) - P_{at}(t)) + rP_{ag}(t) + rP_{at}(t) \\ &= -4rP_{ac}(t) + r. \end{aligned}$$

The solution to the differential equation is

$$P_{ac}(t) = \frac{1}{4} - \frac{1}{4}e^{-4rt}.$$

This is the common term for all the non-diagonal entries of $\boldsymbol{P}(t)$.

7.9 a) Consider

$$\begin{aligned} (\pi Q)_a = \sum_i \pi_i Q_{ia} &= p_a(-\alpha(1 - p_a) + p_g\alpha p_a + p_c\alpha p_a + p_t\alpha p_a) \\ &= p_a(-\alpha(1 - p_a) + \alpha p_g + \alpha p_c + \alpha p_t) \\ &= p_a(-\alpha + \alpha(p_a + p_g + p_c + p_t)) = p_a(-\alpha + \alpha) = 0. \end{aligned}$$

Similarly for $(\pi Q)_j$, for $j \in \{g, c, t\}$.

b) $P_{ac}(1.5) = (1 - e^{-\alpha t})p_c = (1 - e^{-2(1.5)})(0.207) = 0.1967$.

7.10 Consider

$$\begin{aligned} (\pi Q)_a &= -(\alpha p_g + \beta p_r)p_a + \alpha p_a p_g + \beta p_a p_c + \beta p_a p_t \\ &= p_a(\beta p_c + \beta p_t - \beta p_r) = p_a \beta(0) = 0. \end{aligned}$$

Similarly for $(\pi Q)_j$, for $j \in \{g, c, t\}$.

7.11

$$\begin{aligned} \frac{d}{dt}e^{t\boldsymbol{A}} &= \frac{d}{dt}\sum_{n=0}^{\infty} \frac{t^n}{n!}\boldsymbol{A}^n = \sum_{n=0}^{\infty} \frac{1}{n!}\boldsymbol{A}^n \frac{d}{dt}t^n = \sum_{n=1}^{\infty} \frac{1}{n!}\boldsymbol{A}^n n t^{n-1} \\ &= \boldsymbol{A}\sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)}\boldsymbol{A}^{n-1} = \boldsymbol{A}\sum_{n=0}^{\infty} \frac{t^n}{n!}\boldsymbol{A}^n = \boldsymbol{A}e^{t\boldsymbol{A}}. \end{aligned}$$

Similarly, $\frac{d}{dt}e^{t\boldsymbol{A}} = e^{t\boldsymbol{A}}\boldsymbol{A}$.

7.12 Write $\boldsymbol{A} = \boldsymbol{SDS}^{-1}$, where

$$\boldsymbol{D} = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_k \end{pmatrix}.$$

Then,

$$\begin{aligned} \det e^{\boldsymbol{A}} &= \det\left(\boldsymbol{S}e^{\boldsymbol{D}}\boldsymbol{S}^{-1}\right) = (\det \boldsymbol{S})\left(\det e^{\boldsymbol{D}}\right)\left(\det \boldsymbol{S}^{-1}\right) \\ &= (\det \boldsymbol{S})\left(\prod_{i=1}^{k} e^{\lambda_i}\right)(\det \boldsymbol{S})^{-1} = e^{\sum_{i=1}^{k}\lambda_i} = e^{\mathrm{tr}\ \boldsymbol{A}}. \end{aligned}$$

7.13 Taking limits on both sides of $\boldsymbol{P}'(t) = \boldsymbol{P}(t)\boldsymbol{Q}$, as $t \to \infty$, gives that $\boldsymbol{0} = \boldsymbol{\pi Q}$. This uses the fact that if a differentiable function $f(t)$ converges to a constant then the derivative $f'(t)$ converges to 0.

7.14 a)

$$\boldsymbol{Q} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -4 & 4 & 0 \\ 1 & -7 & 6 \\ 6 & 2 & -8 \end{pmatrix} \end{array}.$$

b) $\boldsymbol{\pi} = (11/25, 8/25, 6/25)$.

c)

$$\widetilde{P} = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 \\ 1/7 & 0 & 6/7 \\ 3/4 & 1/4 & 0 \end{pmatrix} \end{array}.$$

d) $\boldsymbol{\psi} = (11/37, 14/37, 12/37)$.

7.15 a)

$$\boldsymbol{Q} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 0 & 0 & -1 \end{pmatrix} \end{array}.$$

b)

$$\frac{1}{4}\begin{pmatrix} 1+e^{-2t}+2e^{-t}\cos t & 1-e^{-2t}+2e^{-t}\sin t & 1+e^{-2t}-2e^{-t}\cos t & 1-e^{-2t}-2e^{-t}\sin t \\ 1-e^{-2t}-2e^{-t}\sin t & 1+e^{-2t}+2e^{-t}\cos t & 1-e^{-2t}+2e^{-t}\sin t & 1+e^{-2t}-2e^{-t}\cos t \\ 1+e^{-2t}-2e^{-t}\cos t & 1-e^{-2t}-2e^{-t}\sin t & 1+e^{-2t}+2e^{-t}\cos t & 1-e^{-2t}+2e^{-t}\sin t \\ 1-e^{-2t}+2e^{-t}\sin t & 1+e^{-2t}-2e^{-t}\cos t & 1-e^{-2t}-2e^{-t}\sin t & 1+e^{-2t}+2e^{-t}\cos t \end{pmatrix}$$

7.16 For the Yule process started with $i = 1$ individual, the transition function is

$$P_{1j}(t) = e^{-\lambda t}\left(1 - e^{-\lambda t}\right)^{j-1}, \text{ for } j \geq 1,$$

which is a geometric distribution with parameter $e^{-\lambda t}$. The Yule process started with i individuals is equivalent to i independent Yule processes each started with one individual. The result follows since the sum of i independent geometric random variables with parameter p has a negative binomial distribution with parameters i and p, and probability mass function

$$P_{ij}(t) = \binom{j-1}{i-1} e^{-\lambda i t} \left(1 - e^{-\lambda t}\right)^{j-i}, \text{ for } j \geq i.$$

7.17 This is a Yule process. The distribution of X_8, the size of the population at $t = 8$, is negative binomial with parameters $r = 4$ and $p = e^{-(1.5)8} = e^{-12}$. Mean and variance are

$$E(X_8) = \frac{r}{p} = \frac{4}{e^{-12}} = 651019 \text{ and } Var(X_8) = \frac{r(1-p)}{p^2} = 1.059558 \times 10^{11}.$$

7.18 a) The probability that the transition from i is a birth is $\lambda_i/(\lambda_i + \mu_i)$. Conditioning on the first transition,

$$E[T_i] = \frac{\lambda_i}{\lambda_i + \mu_i}\left(\frac{1}{\lambda_i}\right) + \frac{\mu_i}{\lambda_i + \mu_i}\left(E[T_{i-1}] + E[T_i]\right).$$

Solving for $E[T_i]$ gives the result.

b)

$$E[T_i] = \frac{1 - (\mu/\lambda)^{i+1}}{\lambda - \mu}, \text{ for } i = 0, 1, \ldots.$$

7.19 Let X_t denote the number of taxis at the stand at time t. The process is an M/M/1 queue. The stationary distribution is geometric with parameter $1 - \lambda/\mu$. The desired probability is $1 - \pi_0 = 1 - (1 - \lambda/\mu) = \lambda/\mu$.

7.20 The M/M/∞ queue is a birth-and-death process with $\lambda_i = \lambda$ and $\mu_i = i\mu$. The stationary distribution is

$$\pi_k = \pi_0 \prod_{i=1}^{k} \frac{\lambda}{i\mu} = \pi_0 \frac{(\lambda/\mu)^k}{k!}, \text{ for } k = 1, 2, \ldots.$$

It follows that the stationary distribution is Poisson with parameter λ/μ. The latter is the mean number of customers in the system.

7.21 Make 4 an absorbing state. We have

$$(-\boldsymbol{v})^{-1} = \left(-\begin{pmatrix} -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ 0 & 2 & -3 & 1 \\ 0 & 0 & 3 & -4 \end{pmatrix}\right)^{-1} = \begin{pmatrix} 10 & 9 & 4 & 1 \\ 9 & 9 & 4 & 1 \\ 8 & 8 & 4 & 1 \\ 6 & 6 & 3 & 1 \end{pmatrix},$$

with row sums $(24, 23, 21, 16)$. The desired mean time is 23.

7.22 Model the process as an absorbing chain with generator

$$\boldsymbol{Q} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & C \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ C \end{matrix} & \begin{pmatrix} -1/10 & 1/10 & 0 & 0 & 0 \\ 0 & -1/20 & 1/20 & 0 & 0 \\ 0 & 0 & -1/30 & 1/30 & 0 \\ 0 & 0 & 0 & -1/40 & 1/40 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix},$$

where C denotes completing the exam. This gives

$$\boldsymbol{P}(45) = e^{45\boldsymbol{Q}} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & C \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ C \end{matrix} & \begin{pmatrix} 0.0111 & 0.1886 & 0.3884 & 0.2824 & 0.1296 \\ 0.0 & 0.1054 & 0.3532 & 0.3413 & 0.2002 \\ 0.0 & 0.0 & 0.2231 & 0.4061 & 0.3708 \\ 0.0 & 0.0 & 0.0 & 0.3247 & 0.6753 \\ 0.0 & 0.0 & 0.0 & 0.0 & 1.0 \end{pmatrix} \end{matrix}.$$

The desired probabilities are a) $P_{1C}(45) = 0.1296$, and b) $P_{13}(45) = 0.3884$.

7.23 a)

$$\boldsymbol{P} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} -1/2 & 1/2 & 0 & 0 & 0 \\ 1/10 & -3/5 & 1/2 & 0 & 0 \\ 0 & 1/5 & -7/10 & 1/2 & 0 \\ 0 & 0 & 3/10 & -11/20 & 1/4 \\ 0 & 0 & 0 & 2/5 & -2/5 \end{pmatrix} \end{matrix}.$$

b) Solving $\boldsymbol{\pi Q} = \boldsymbol{0}$ gives the stationary distribution

$$\boldsymbol{\pi} = (0.0191, 0.0955, 0.2388, 0.3979, 0.2487).$$

The long -term expected number of working machines is

$$0\pi_0 + 1\pi_1 + 2\pi_2 + 3\pi_3 + 4\pi_4 = 2.76.$$

c) The desired probability, using numerical software, is

$$P_{42}(5) = \left(e^{5\boldsymbol{Q}}\right)_{42} = 0.188.$$

7.24 The process is a birth-and-death process with $\lambda_i = \lambda/(i+1)$ and $\mu_i = 1/\alpha$. For the stationary distribution

$$\pi_0 = \left(\sum_{k=0}^{\infty} \prod_{i=1}^{k} \frac{\lambda\alpha}{i}\right)^{-1} = \left(\sum_{k=0}^{\infty} (\lambda\alpha)^k \frac{1}{k!}\right)^{-1} = e^{-\lambda\alpha}$$

and

$$\pi_k = \frac{e^{-\lambda\alpha}(\lambda\alpha)^k}{k!}, \text{ for } k = 0, 1, \ldots.$$

The distribution is Poisson with parameter $\lambda\alpha$. a) The long-term average number of people in line is $\lambda\alpha$. b) The desired probability is $1 - (1 + \lambda\alpha)e^{-\lambda\alpha}$.

7.25 a) $\boldsymbol{\psi} = (0.1, 0.3, 0.3, 0.3)$.

b) We have that $(q_1, q_2, q_3, q_4) = (1, 1/2, 1/3, 1/4)$. The stationary distribution $\boldsymbol{\pi}$ is proportional to $(0.1, 0.3(2), 0.3(3), 0.3(4))$. This gives

$$\boldsymbol{\psi} = \frac{1}{2.8}(0.1, 0.6, 0.9, 1.2) = (0.036, 0.214, 0.321, 0.428).$$

7.26 The process has generator

$$\boldsymbol{Q} = \begin{matrix} & \begin{matrix} 0 & 1 & 2 & 3 \end{matrix} \\ \begin{matrix} 0 \\ 1 \\ 2 \\ 3 \end{matrix} & \begin{pmatrix} -3/6 & 3/6 & 0 & 0 \\ 1/24 & -9/24 & 2/6 & 0 \\ 0 & 2/24 & -6/24 & 1/6 \\ 0 & 0 & 3/24 & -3/24 \end{pmatrix} \end{matrix}.$$

The stationary distribution is $\boldsymbol{\pi} = (1/125, 12/125, 48/125, 64/125)$. The desired probability is $\pi_3 = 64/125 = 0.512$.

7.27 a) If the first dog has i fleas, then the number of fleas on the dog increases by one the first time that one of the $N - i$ fleas on the other dog jumps. The time of that jump is the minimum of $N - i$ independent exponential random variables with parameter λ. Similarly, the number of fleas on the first dog decreases by one when one of the i fleas on that dog first jumps.

b) The local balance equations are $\pi_i(N - i)\lambda = \pi_{i+1}(i + 1)\lambda$. The equations are satisfied by the stationary distribution

$$\pi_k = \binom{N}{k}\left(\frac{1}{2}\right)^k, \text{ for } k = 0, 1, \ldots, N,$$

which is a binomial distribution with parameters N and $p = 1/2$.

c) 0.45 minutes.

7.28 A linear birth-and-death process with immigration (see Table **??**) has parameters $\lambda = 3$, $\mu = 4$, and $\alpha = 2$. Find the stationary distribution.

$$\pi_k = \frac{2 \cdot 5 \cdots 3(k-1)}{4^k k!}\left(\frac{1}{4}\right)^{2/3}, \text{ for } k = 0, 1, \ldots.$$

7.29 Consider an absorbing, continuous-time Markov chain with possibly more than one absorbing states.

a) Argue that the continuous-time chain is absorbed in state a if and only if the embedded discrete-time chain is absorbed in state a.

b) Let

$$\boldsymbol{Q} = \begin{matrix} & \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{matrix} & \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & -3 & 2 & 0 & 0 \\ 0 & 2 & -4 & 2 & 0 \\ 0 & 0 & 2 & -5 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{matrix}$$

be the generator matrix for a continuous-time Markov chain. For the chain started in state 2, find the probability that the chain is absorbed in state 5.

The embedded chain transition matrix, in canonical form, is

$$\widetilde{\boldsymbol{P}} = \begin{array}{c} \\ 1 \\ 5 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 5 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1/3 & 0 & 0 & 2/3 & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 3/5 & 0 & 2/5 & 0 \end{pmatrix} \end{array}.$$

By the discrete-time theory for absorbing Markov chains, write

$$\widetilde{\boldsymbol{Q}} = \begin{pmatrix} 0 & 2/3 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 2/5 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{\boldsymbol{R}} = \begin{pmatrix} 1/3 & 0 \\ 0 & 0 \\ 0 & 3/5 \end{pmatrix}.$$

The matrix of absorption probabilities is

$$(\boldsymbol{I} - \widetilde{\boldsymbol{Q}})^{-1}\widetilde{\boldsymbol{R}} = \begin{array}{c} \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{cc} 1 & 5 \end{array} \\ \begin{pmatrix} 4/7 & 3/7 \\ 5/14 & 9/14 \\ 1/7 & 6/7 \end{pmatrix} \end{array}.$$

The desired probability is 3/7.

7.30 a) The process is an M/M/1 queue with $\lambda = 2$ and $\mu = 3$. The stationary distribution is a geometric distribution with parameter $p = 1-\lambda/\mu = 1/3$. The mean is $(1-p)/p = 2$.

b) The desired probability is

$$P(Y > 3) = 1 - P(Y \leq 2) = 1 - (1/3) - (1/3)(2/3) - (1/3)(2/3)^2 = 0.346.$$

7.31 The process is an M/M/2 queue with $\lambda = 2$, $\mu = 3$, and $c = 2$. See the solution in Example 7.25. The desired probability is

$$\pi_0 = \left(1 + \frac{2}{3} + \frac{1}{3}\right)^{-1} = \frac{1}{2}$$

7.32 a) The process is an M/M/3 queueing system with $\lambda = 15$, $\mu = 6$, and $c = 3$. The expected number of callers in the queue is

$$L_q = \frac{\pi_0}{3!}\left(\frac{15}{6}\right)^3 \frac{15}{18}\left(\frac{1}{1 - 15/18}\right)^2,$$

where

$$\pi_0 = \left(1 + \frac{15}{6} + \frac{1}{2}\left(\frac{15}{6}\right)^2 + \frac{(15/6)^3}{3!}\left(\frac{1}{1 - 15/18}\right)\right)^{-1} = \frac{4}{89} = 0.045.$$

This gives $L_q = 3.511$.

b) By Little's formula, the expected waiting time is $W_q = L_q/\lambda = 3.511/15 = 0.234$, or about 14 minutes.

7.33 a) The long-term expected number of customers in the queue L is the mean of a geometric distribution on $0, 1, 2, \ldots$, with parameter $1 - \lambda/\mu$, which is $\lambda/(\mu - \lambda)$. If both λ and μ increase by a factor of k, this does not change the value of L.

b) The expected waiting time is $W = L/\lambda$. The new waiting time is $L/(k\lambda) = W/k$.

7.34 Using Little's formula, $L \approx 20$ and $W \approx 3$. Thus $\lambda = L/W \approx 20/3$ customers per minute, or 400 customers per hour. The shop is open 16 hours per day. The estimated total revenue is $400 \times 16 \times 4 = \$25,600$ per day.

7.35 With $\lambda = 4$,

$$\boldsymbol{R} = \begin{pmatrix} 3/4 & 0 & 1/4 \\ 1/4 & 1/2 & 1/4 \\ 1/4 & 3/4 & 0 \end{pmatrix}.$$

7.36

$$\boldsymbol{P}(t) = \begin{pmatrix} p + qe^{-\lambda t} & q - qe^{-\lambda t} \\ p - pe^{-\lambda t} & q + pe^{-\lambda t} \end{pmatrix}.$$

7.37 a) Choose N such that $P(Y > N) < 0.5 \times 10^{-3}$, where Y is a Poisson random variable with parameter $9 \times 0.8 = 7.2$. This gives $N = 17$.

b) With $\lambda = 9$,

$$\boldsymbol{R} = \frac{1}{9}\boldsymbol{Q} + \boldsymbol{I} = \begin{pmatrix} 5/9 & 1/9 & 2/9 & 1/9 \\ 2/9 & 2/3 & 0 & 1/9 \\ 1/3 & 1/3 & 0 & 1/3 \\ 4/9 & 2/9 & 0 & 1/3 \end{pmatrix}.$$

This gives

$$\begin{aligned} \boldsymbol{P}(0.8) &\approx \sum_{k=0}^{17} \boldsymbol{R}^k \frac{e^{-9(0.8)}(9(0.8))^k}{k!} \\ &= \begin{pmatrix} 0.410136 & 0.326706 & 0.0927968 & 0.169861 \\ 0.395769 & 0.352768 & 0.0858808 & 0.165082 \\ 0.404007 & 0.336012 & 0.0898739 & 0.169608 \\ 0.407465 & 0.330164 & 0.0906602 & 0.171212 \end{pmatrix} \end{aligned}$$

c)

```
> q = matrix(c(-4,1,2,1,2,-3,0,1,3,3,-9,3,4,2,0,-6),nrow=4,byrow=T)
> expm(q*0.8)
          [,1]      [,2]       [,3]      [,4]
[1,] 0.4103385 0.3268743 0.09284172 0.1699455
[2,] 0.3959714 0.3529367 0.08592572 0.1651661
[3,] 0.4042088 0.3361800 0.08991882 0.1696925
[4,] 0.4076668 0.3303323 0.09070507 0.1712959
```

7.38 `R`: Tom's exam.

```
> trials <- 100000
> sim1 <- numeric(trials)
> sim2 <- numeric(trials)
> for (i in 1:trials) {
+ times <- cumsum(c(rexp(1,1/10),rexp(1,1/20),rexp(1,1/30),rexp(1,1/40)))
+ sim1[i] <- if (times[4] < 45) 1 else 0
+ sim2[i] <- if (times[2] < 45 & times[3] > 45) 1 else 0
+ }
> mean(sim1)
[1] 0.1303
> mean(sim2)
[1] 0.38718
```

7.39 R: Simulate an M/M/∞ queue.

```
> trials <- 10000
> sim <- numeric(trials)
> lambda <- 2
> mu <- 1
> for (i in 1:trials) {
+ serv <- 0
+ t <- 0
+ for (k in 1:100) {  # simulate for 100 transitions
+  arr <- rexp(1,lambda)
+  if (serv ==0) { t <- t + arr
+  serv <- 1} else {
+  s <- rexp(serv,mu)
+  newt <- min(arr,s)
+  if (newt==arr) serv <- serv + 1 else serv <- serv - 1
+  t <- t + newt
+  }   }
+  sim[i] <- serv
+  }
> mean(sim)
[1] 2.496
```

7.40 a)

$$\boldsymbol{Q} = \begin{pmatrix} -0.055 & 0.055 & 0 & 0 & 0 & 0 & 0 \\ 0.008 & -0.068 & 0.060 & 0 & 0 & 0 & 0 \\ 0 & 0.008 & -0.047 & 0.039 & 0 & 0 & 0 \\ 0 & 0 & 0.008 & -0.047 & 0.033 & 0 & 0.006 \\ 0 & 0 & 0 & 0.009 & -0.045 & 0.029 & 0.007 \\ 0 & 0 & 0 & 0 & 0.002 & -0.044 & 0.042 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

The fundamental matrix is

$$\boldsymbol{F} = \begin{pmatrix} 21.23 & 20.93 & 32.01 & 31.06 & 23.46 & 15.46 \\ 3.05 & 20.93 & 32.01 & 31.06 & 23.46 & 15.46 \\ 0.62 & 4.27 & 32.01 & 31.06 & 23.46 & 15.46 \\ 0.12 & 0.85 & 6.37 & 31.05 & 23.46 & 15.46 \\ 0.03 & 0.18 & 1.31 & 6.40 & 27.73 & 18.27 \\ 0.00 & 0.01 & 0.06 & 0.29 & 1.26 & 23.56 \end{pmatrix},$$

with row sums $(144.15, 125.97, 106.88, 77.32, 153.91, 25.18)$, which gives the mean absorption times, in months, from each state.

b) The desired probabilities are $P_{17}(t)$, for $t = 60, 120, 18, 240$, which gives, respectively, $0.090, 0.434, 0.739, 0.900$.

Chapter 8

8.1
$$\frac{\partial f}{\partial t} = \frac{e^{-x^2/2t}x^2}{2\sqrt{2\pi}t^{5/2}} - \frac{e^{-x^2/2t}}{2\sqrt{2\pi}t^{3/2}},$$

and
$$\frac{1}{2}\frac{\partial f}{\partial x^2} = \frac{-1}{2}\frac{\partial}{\partial x}\frac{e^{-x^2/2t}x}{\sqrt{2\pi}t^{3/2}} = \frac{e^{-x^2/2t}x^2}{2\sqrt{2\pi}t^{5/2}} - \frac{e^{-x^2/2t}}{2\sqrt{2\pi}t^{3/2}}.$$

8.2 a) $P(B_2 \le 1) = \int_{-\infty}^{1} \frac{1}{\sqrt{4\pi}} e^{-x^2/4}\, dx = 0.760.$

b) $E(B_4|B_1 = x) = E(B_4 - B_1 + x|B_1 = x) = E(B_4 - B_1) + x = E(B_3) + x = x.$

c)
$$\text{Cor}(B_{t+s}, B_s) = \frac{\text{Cov}(B_{t+s}B_s}{SD(B_{t+s})SD(B_s)} = \frac{s}{\sqrt{t+s}\sqrt{s}} = \sqrt{\frac{s}{t+s}}.$$

d) $Var(B_4|B_1) = Var(B_4 - B_1 + B_1) = Var(B_4 - B_1) = Var(B_3) = 3.$

e)
$$P(B_3 \le 5|B_1 = 2) = P(B_3 - B_1 \le 3|B_1 = 2) = P(B_3 - B_1 \le 3) = P(B_2 \le 3)$$
$$= \int_{\infty}^{3} \frac{1}{\sqrt{4\pi}} e^{-x^2/4}\, dx = 0.983.$$

8.3 a) Write $X_1 + X_2 = (B_1 - 3) + (B_2 - 3) = B_1 + B_2 - 6$, which is normally distributed with mean -6 and variance
$$Var(B_1) + Var(B_2) + 2\text{Cov}(B_1, B_2) = 1 + 2 + 2(1) = 5.$$

This gives
$$P(X_1 + X_2 > -1) = \int_{-1}^{\infty} \frac{1}{\sqrt{10\pi}} e^{-(x+6)^2/10}\, dx = 0.013.$$

b) The conditional distribution of X_2 given $X_1 = 0$, is the same as that of $X_2 - X_1 = B_2 - B_1$, which is the same as that of B_1. The conditional density is
$$f_{X_2|X_1}(x|0) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \text{ for } -\infty < x < \infty.$$

c)
$$\text{Cov}(X_3, X_4) = \text{Cov}(X_3, X_4 - X_3 + X_3) = \text{Cov}(X_3, X_4 - X_3) + Var(X_3) = Var(X_3) = 3.$$

d) $E(X_4|X_1) = E(X_4 - X_1 + X_1|X_1) = E(X_4 - X_1) + X_1 = X_1.$

8.4 The desired probability is
$$P(X_1 > 0|X_{3/4} = \sigma/2) = P(X_1 - X_{3/4} > -\sigma/2)) = P(X_{1/4} > -\sigma/2).$$

The random variable $X_{1/4}$ is normally distributed with mean 0 and variance $\sigma^2/4$. This gives
$$P(X_{1/4} > -\sigma/2) = P(X_{1/4}/(\sigma/2) > -1) = P(Z > -1) = 0.841.$$

where Z has a standard normal distribution.

8.5 That the conditional distribution is normal is immediate from the fact that Brownian motion is a Gaussian process. For the joint density of B_s and B_t, since

$$\{B_s = x, B_t = y\} = \{B_s = x, B_t - B_s = y - x\},$$

it follows that the joint density is

$$\begin{aligned} f_{B_s,B_t}(x, y) = f_{B_s}(x) f_{B_{t-s}}(y - x) &= \frac{1}{\sqrt{2\pi s}} e^{-x^2/2s} \frac{1}{\sqrt{2\pi(t-s)}} e^{-(y-x)^2/2(t-s)} \\ &= Ce^{-(x-sy/t)^2/(2s(t-s)/t)}, \end{aligned}$$

for some constant C. This also shows that the conditional distribution is normal with $E(B_s|B_t = y) = sy/t$ and $Var(B_s|B_t = y) = s(t - s)/t$.

8.6 The process is a Gaussian process with continuous paths. The mean function is equal to 0, for all t. The covariance function is

$$\begin{aligned} E\left((B_{t+s} - B_s)(B_{u+s} - B_s)\right) &= E(B_{t+s}B_{u+s} - B_{t+s}B_s - B_sB_{u+s} + B_s^2) \\ &= E(B_{t+s}B_{u+s}) - E(B_{t+s}B_s) - E(B_sB_{u+s}) + E(B_s^2) \\ &= \min\{t + s, u + s\} - 2s + s \\ &= \min\{t, u\} + s - 2s + s = \min\{t, u\}. \end{aligned}$$

Thus the translation process is a standard Brownian motion.

8.7 One checks that the reflection is a Gaussian process with continuous paths. Further, the mean function is $E(-B_t) = 0$ and the covariance function is $E((-B_s)(-B_t)) = E(B_sB_t) = \min\{s, t\}$.

8.8

$$\text{Cov}(X_s, X_t) = \text{Cov}(\mu s + \sigma B_s, \mu t + \sigma B_t) = \sigma^2 \text{Cov}(B_s, B_t) = \sigma^2 \min\{s, t\}.$$

8.9 Let $W_t = B_{2t} - B_t$, where $(B_t)_{t\geq 0}$ is standard Brownian motion. The process is a Gaussian process, but not a Brownian motion process, as it does not have independent increments. For instance,

$$\begin{aligned} E(W_4(W_6 - W_5)) &= E((B_8 - B_4)(B_{12} - B_6 - B_{10} + B_5)) \\ &= E(B_8B_{12} - B_8B_6 - B_8B_{10} + B_8B_5 - B_4B_{12} + B_4B_6 + B_4B_{10} - B_4B_5) \\ &= 8 - 6 - 8 + 5 - 4 + 4 + 4 - 4 = -1 \neq 0. \end{aligned}$$

8.10

$$E(X_t) = E(B_t - t(B_1 - y)) = E(B_t) - t(E(B_1) - y) = x - t(x - y), \text{ for } 0 \leq t \leq 1.$$

8.11 By translation, the desired probability is $P(B_2 > 1|B_3 = 0)$. The distribution of B_2 given $B_3 = 0$ is normal with mean 0 and variance 2/3. In R,

```
> 1-pnorm(1,0,sqrt(2/3))
[1] 0.1103357
```

8.12 Let $(X_t)_{t\geq 0}$ be a Brownian motion with drift μ and variance σ^2. For $s, t \geq 0$,

$$X_{s+t} - X_t = \mu(t+s) + \sigma^2 B_{s+t} - \mu_t - \sigma^2 B_t = \mu s + \sigma^2(B_{s+t} - B_t),$$

which has the same distribution as $\mu s + \sigma^2 B_s = X_s$. Thus, Brownian motion with drift has stationary increments.

For $0 \leq q < r \leq s < t$, $X_r - X_q = \mu(r-q)\sigma^2 B_{r-q}$. Also, $X_t - X_s = \mu(t-s)\sigma^2 B_{t-s}$. Independent increments follows from independent increments for standard Brownian motion.

8.13

$$\begin{aligned} E(X_s X_t) &= E(X_s(X_t - X_s + X_s) = E(X_s X_{t-s}) + E(X_s^2) \\ &= E(X_s)E(X_{t-s}) + E(X_s^2) = \mu s \mu(t-s) + (\sigma^2 s - \mu^2 s^2) \\ &= st\mu^2 + \sigma^2 s. \end{aligned}$$

8.14 Write $X_t = 1.5 - t + 2B_t$. This gives

$$P(X_3 > 0) = P(1.5 - 3 + 2B_3 > 0) = P(B_3 > 0.75).$$

In R,

```
> 1-pnorm(0.75,0,sqrt(3))
[1] 0.3325028
```

8.15 Results for the first three quarters suggest stationary increments. However, this is not supported by the fourth quarter data.

8.16 Since the difference of independent normal random variables is normal, it follows that $(Z_t)_{t\geq 0}$ is a Gaussian process. The covariance function is

$$\begin{aligned} \mathrm{Cov}(Z_s, Z_t) &= \mathrm{Cov}(a(X_s - Y_s), a(X_t - Y_t)) \\ &= a^2 \left(\mathrm{Cov}(X_s, X_t) - \mathrm{Cov}(X_s, Y_t) - \mathrm{Cov}(Y_s, X_t) + \mathrm{Cov}(Y_s, Y_t)\right) \\ &= a^2 2 \min(s,t). \end{aligned}$$

To obtain a standard Brownian motion requires $a = \pm 1/\sqrt{2}$.

8.17 Let $Z = 1/X^2$. For $t > 0$,

$$\begin{aligned} P(Z \leq t) &= P(1/X^2 \leq t) = P(X^2 \geq 1/t) \\ &= P(X \geq 1/\sqrt{t}) + P(X \leq -1/\sqrt{t}) = 2P(X \leq -1/\sqrt{t}). \end{aligned}$$

Differentiating with respect to t,

$$\begin{aligned} f_Z(t) &= t^{-3/2} f_X(-1/\sqrt{t}) = t^{-3/2} \frac{1}{\sqrt{2\pi/a^2}} e^{-1/(2t/a^2)} \\ &= \frac{|a|}{\sqrt{2\pi t^3}} e^{-a^2/2t}, \text{ for } t > 0, \end{aligned}$$

which is the density function of the first hitting time T_a.

8.18 For $t > 0$, $P(a^2T_1 \le t) = P(T_1 \le t/a^2)$. Differentiating with respect to t gives the density function

$$\frac{1}{a^2} f_{T_1}(t/a^2) = \frac{1}{a^2} \frac{1}{\sqrt{2\pi(t/a^2)^3}} e^{-1/2(t/a^3)} = \frac{|a|}{\sqrt{2\pi t^3}} e^{-a^2/2t} = f_{T_a}(t).$$

8.19

$$E(M_t) = E\left(|B_t|\right) = \int_{-\infty}^{\infty} |x| \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, dx = 2\int_0^{\infty} x \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, dx$$
$$= \sqrt{\frac{2t}{\pi}} \approx (0.80)\sqrt{t}.$$

For variance, $E(M_t^2) = E\left(B_t^2\right) = t$. This gives

$$Var(M_t) = t - \frac{2t}{\pi} = t\left(1 - \frac{2}{\pi}\right) \approx (0.363)t.$$

8.20 For a Brownian motion path from the origin to $(t, a+b)$, let T_a be the first time the path reaches level a. Create a new path by reflecting the piece on the interval (T_a, t) about the line $y = a$. This creates a Brownian motion path from the origin to $(t, a-b)$, whose maximum value is at least a. The correspondence is invertible, and shows that

$$\{M_t \ge a, B_t \le a - b\} = \{B_t \ge a + b\},$$

from which the result follows.

8.21 a)

$$P(X_t = a) = P(T_a \le t) = \frac{2}{\sqrt{2\pi t}} \int_a^{\infty} e^{-x^2/2t}\, dx.$$

b) For $x < a$,

$$P(X_t \le x) = P(B_t \le x, M_t < a) = P(B_t \le x) - P(B_t \le x, M_t \ge a)$$
$$= P(B_t \le x) - P(B_t \le x, M_t \ge (a - (a - x))).$$

By Exercise 8.16,

$$P(X_t \le x) = P(B_t \le x) - P(B_t \ge a + (a - x)) = P(B_t \le x) - P(B_t \ge 2a - x),$$

and the result follows.

8.22 The event $\{Z \le z\}$ is equal to the event that there is at least one zero in $[t, z]$. The result follows by Theorem 8.1.

8.23 a) Let $A_{a,b}$ denote the event that standard Brownian motion is not zero in (a, b). Then

$$P(A_{r,t}|A_{r,s}) = \frac{P(A_{r,t})}{P(A_{r,s})} = \frac{1 - P(\text{at least 1 zero in } (r,t))}{1 - P(\text{at least 1 zero in } (r,s))}$$
$$= \frac{1 - \frac{2}{\pi}\arccos\sqrt{r/t}}{1 - \frac{2}{\pi}\arccos\sqrt{r/s}} = \frac{\arcsin\sqrt{r/t}}{\arcsin\sqrt{r/s}}.$$

b) Take the limit, as $r \to 0$, in the result in a). By l'Hospital's rule, the desired probability is

$$\lim_{r\to 0}\frac{\arcsin\sqrt{r/t}}{\arcsin\sqrt{r/s}} = \lim_{r\to 0}\frac{1/(2\sqrt{1-r/t}\sqrt{tr})}{1/(2\sqrt{1-r/s}\sqrt{sr})} = \lim_{r\to 0}\frac{\sqrt{1-r/s}\sqrt{s}}{\sqrt{1-r/t}\sqrt{t}} = \sqrt{\frac{s}{t}}.$$

8.24 Derive the mean and variance of geometric Brownian motion.

Let $G_t = G_0 e^{X_t}$, where $X_t = \mu t + \sigma B_t$. Then

$$\begin{aligned}E(G_t) &= E(G_0 e^{X_t}) = G_0 E(e^{\mu t+\sigma B_t}) = G_0 e^{\mu t} E(e^{\sigma B_t})\\ &= G_0 e^{\mu t}\int_{-\infty}^{\infty} e^{\sigma x}\frac{1}{\sqrt{2\pi t}}e^{-x^2/2t}\,dx\\ &= G_0 e^{t(\mu+\sigma^2/2)}.\end{aligned}$$

Similarly,

$$Var(G_t) = G_0^2 e^{2t(\mu+\sigma^2/2)}\left(e^{t\sigma^2}-1\right).$$

8.25 Let G_t denote the price of the stock after t years. Then

$$\begin{aligned}P(G_1 \geq 40) &= P\left(G_0 e^{X_1} \geq 40\right) = P\left(35e^{-0.25+(0.4)B_1} \geq 40\right)\\ &= P\left(-0.25+(0.4)B_1 \geq \ln\frac{40}{35}\right) = P\left(B_1 \geq \frac{0.25+\ln 40/35}{0.4}\right)\\ &= P(B_1 \geq 0.9588) = 0.1688.\end{aligned}$$

8.26 In the notation of Example 8.16, $G_0 = 35$, $K = 40$, $t = 1$, $\mu = -0.25$ and $\sigma = 0.4$. The expected payoff of the option is

$$G_0 e^{t(\mu+\sigma^2/2)} P\left(Z > \frac{\beta-\sigma t}{\sqrt{t}}\right) - KP\left(Z > \frac{\beta}{\sqrt{t}}\right),$$

where $\beta = (\ln(K/G_0) - \mu t)/\sigma = (\ln(40/35)+0.25)/0.4 = 0.9588$. The expected payoff is

$$35e^{-0.25+0.08}P(Z > 0.9588-0.4) - 40P(Z > 0.9588) = 1.755.$$

8.27 Write $Z_{n+1} = \sum_{i=1}^{Z_n} X_i$. Then,

$$\begin{aligned}E\left(\frac{Z_{n+1}}{\mu^{n+1}}|Z_0,\ldots,Z_n\right) &= \frac{1}{\mu^{n+1}}E\left(\sum_{i=1}^{Z_n} X_i|Z_0,\ldots,Z_n\right)\\ &= \frac{1}{\mu^{n+1}}E\left(\sum_{i=1}^{Z_n} X_i|Z_n\right)\\ &= \frac{1}{\mu^{n+1}}Z^n\mu = \frac{Z_n}{\mu^n}.\end{aligned}$$

8.28 Polya's Urn.

Observe that

$$E(X_{n+1}|X_0,\ldots,X_n) = X_n + \frac{X_n}{n+2}.$$

This gives

$$\begin{aligned} E(R_{n+1}|X_0,\ldots,X_n) &= E\left(\frac{X_{n+1}}{n+3}|X_0,\ldots,X_n\right) \\ &= \frac{1}{n+3}\left(X_n + \frac{X_n}{n+2}\right) \\ &= \frac{X_n}{n+2} = R_n. \end{aligned}$$

8.29

$$\begin{aligned} E(B_t^3 - 3tB_t|B_r, 0 \le r \le s) &= E((B_t - B_s + B_s)^3|B_r, 0 \le r \le s) - 3tE(B_t|B_r, 0 \le r \le s) \\ &= E((B_t - B_s)^3 + 3B_sE(B_t - B_s)^2 + 3B_s^2E(B_t - B_s) + B_s^3 - 3tB_s \\ &= E(B_{t-s}^3) + 3B_sE(B_{t-s}^2) + 3B_s^2E(B_{t-s}) + B_s^3 - 3tB_s \\ &= 0 + 3B_s(t-s) + 0 + B_s^3 - 3tB_s \\ &= B_s^3 - 3sB_s. \end{aligned}$$

Also,

$$E(|B_t^3 - 3tB_t|) \le E(|B_t^3|) + 3tE(|B_t|) < \infty.$$

8.30 Let T be the first time level $a = 3$ is hit. By Examples 8.26 and 8.27,

$$E(T) = a/\mu = 3/2 = 1.5 \text{ and } Var(T) = a\sigma^2/\mu^3 = 3(4)/8 = 1.5.$$

Let $\overline{X}$ denote the average of the 25 runs. By the central limit theorem,

$$P(1 \le \overline{X} \le 2) = P\left(\frac{1-1.5}{\sqrt{1.5/25}} \le \overline{X} \le \frac{2-1.5}{\sqrt{1.5/25}}\right) \approx P(-2.04 \le Z \le 2.04) = 0.978.$$

8.31 a) By translation the probability is equal to the probability that a standard Brownian motion started at 0 reaches level $a = 5$ before $-b = 4$. By Example 8.22, the desired probability is $b/(a+b) = 4/9$.

b) After suitable translation, as in a), the expectation, as shown in Example 8.23, is $ab = 20$.

8.32

$$\begin{aligned} E(X_t|N_r, 0 \le r \le s) &= E(N_t - t\lambda|N_r, 0 \le r \le s) \\ &= E(N_t - N_s + N_s|N_r, 0 \le r \le s) - t \\ &= E(N_t - N_s) + N_s - t = (t-s) + N_s - t = N_s - s = X_s, \end{aligned}$$

using the fact that $N_t - N_s$ is independent of N_s. Also,

$$E(|X_t|) = E(|N_t - t\lambda|) \le E(N_t + t\lambda) = 2\lambda t < \infty.$$

Thus (X_t) is a martingale.

8.33 Since the optional stopping theorem is satisfied we have that $E(Z_N) = E(Z_0)$. That is,

$$\begin{aligned} 0 = E\left(\sum_{i=1}^{N}(X_i - \mu)\right) &= E\left(\sum_{i=1}^{N} X_i - \sum_{i=1}^{N} \mu\right) \\ &= E\left(\sum_{i=1}^{N} X_i\right) - E(N\mu) = E\left(\sum_{i=1}^{N} X_i\right) - E(N)\mu, \end{aligned}$$

which gives the result.

8.34 a) The desired quantity is $m(t) = \lambda t$.

b) The event $\{T \le t\}$, that level k has been reached by time t, can be determined from knowledge of the process up to time t, that is, by N_t. Thus T is a stopping time. It also satisfies the optional stopping theorem since for $k > 0$,

$$\begin{aligned} P(T < \infty) &= \lim_{t\to\infty} P(T \le t) = \lim_{t\to\infty} P(N_t \ge k) \\ &= \lim_{t\to\infty} 1 - \sum_{i=0}^{k-1} \frac{e^{-\lambda t}(\lambda t)^i}{i!} = 1 - \sum_{i=0}^{k-1} \frac{\lambda^i}{i!} \lim_{t\to\infty} e^{-\lambda t} t^i = 1 - 0 = 1, \end{aligned}$$

and

$$E(|N_t|) = E(N_t) \le k, \text{ for } t < T.$$

Note that $T = S_k$, the kth arrival time.

c)

By the optional stopping theorem,

$$\begin{aligned} 0 = E(M_0) = E(M_T) &= E\left((N_T - \lambda T)^2 - \lambda T\right) \\ &= E(N_T^2) - 2\lambda E(N_T T) + \lambda^2 E(T^2) - \lambda E(T) \\ &= k^2 - 2k\lambda E(T) + \lambda^2 E(T^2) - \lambda E(T) \\ &= k^2 - 2k\lambda(k/\lambda) + \lambda^2 E(T^2) - \lambda(k/\lambda) \\ &= k^2 - 2k^2 + \lambda^2 E(T^2) - k. \end{aligned}$$

This gives $E(T^2) = (k^2 + k)/\lambda^2$. And thus,

$$\text{Var}(T) = E(T^2) - E(T)^2 = (k^2 + k)/\lambda^2 - k^2/\lambda^2 = k/\lambda^2.$$

The standard deviation is $\sqrt{k}/\lambda$.

8.35 a, b) For $a > 0$, let T be the first time that standard Brownian motion (B_t) exits the interval $(-a, a)$. This is the first time the process hits either a or $-a$. By Example 8.23, T is a stopping time which satisfies the optional stopping theorem, and $E(T) = a^2$.

c) The process $M_t = B_t^4 - 6tB_t^2 + 3t^2$ is a martingale. By the optional stopping theorem,

$$0 = E(M_0) = E(M_T) = E(B_T^4 - 6TB_T^2 + 3T^2) = a^4 - 6E(T)a^2 - 3E(T^2) = a^4 - 6a^4 - 3E(T^2).$$

This gives $E(T^2) = (5/3)a^4$, and thus

$$SD(T) = \sqrt{\text{Var}(T)} = \sqrt{(E(T^2) - E(T)^2} = \sqrt{(5/3)a^4 - a^4} = \sqrt{2/3}a^2 \approx (0.82)a^2.$$

8.36 a) For $0 \le s < t$,

$$\begin{aligned} E(Y_t|B_r, 0 \le r \le s) &= E\left(e^{-tc^2/2+cB_t}|B_r, 0 \le r \le s\right) \\ &= e^{-tc^2/2}E\left(e^{c(B_t-B_s+B_s)}|B_r, 0 \le r \le s\right) \\ &= e^{-tc^2/2}E\left(e^{cB_{t-s}}\right)e^{cB_s} \\ &= e^{-tc^2/s}e^{(t-s)c^2/2}e^{cB_s} = e^{-sc^2/2+cB_s} = Y_s. \end{aligned}$$

Thus $(Y_t)_{t\ge 0}$ is a martingale with respect to Brownian motion. The fourth equality is because the moment generating function of a normal random variable with mean 0 and variance $t-s$ is $E(e^{cB_{t-s}}) = e^{(t-s)c^2/2}$.

b) Observe that $B_t = (X_t - \mu t)/\sigma$. This gives

$$E\left(e^{c(X_T-\mu T)/\sigma - c^2T/2}\right) = E\left(e^{cB_T - c^2T/2}\right) = E\left(Y_T\right).$$

By the optional stopping theorem, this is equal to $E(Y_0) = e^{cB_0 - c^2(0)/2} = e^0 = 1$. Let $c = -2\mu/\sigma$. Then,

$$1 = E\left(e^{c(X_T-\mu T)/\sigma - c^2T/2}\right) = E\left(e^{-2\mu X_T/\sigma^2}\right).$$

c) From b),

$$1 = E\left(e^{-2\mu X_T/\sigma^2}\right) = pe^{-2\mu a/\sigma^2} + (1-p)e^{2\mu b/\sigma^2}.$$

Solving for p gives the result.

8.37 a) See Exercise 8.28. Letting $b \to \infty$, gives the probability that the process ultimately reaches level a. The latter event is equal to the event that the maximum value attained is at least a. Since $\mu < 0$, this gives

$$P(M > a) = \lim_{b\to\infty} \frac{1 - e^{2\mu b/\sigma^2}}{e^{-2\mu a/\sigma^2} - e^{2\mu b/\sigma^2}} = e^{2\mu a/\sigma^2} = e^{-2|\mu|a/\sigma^2}, \text{ for } a > 0.$$

It follows that M has an exponential distribution with parameter $2|\mu|/\sigma^2$.

b) By properties of the exponential distribution, mean and standard deviation are $-2|\mu|/\sigma^2 = 2(1.6)/0.4 = 8$.

8.38 a) We have that

$$E(e^{cB_T - c^2T/2}) = E(e^0) = 1.$$

Let $c = \sqrt{2\lambda}$. This gives

$$E\left(e^{\sqrt{2\lambda}B_T}e^{-\lambda T}\right) = 1.$$

b) The first equality is by symmetry. Also, $P(B_T = a) = P(B_T = -a) = 1/2$. This gives

$$P(B_T = a|T < x) = \frac{P(T < x, B_T = a)}{P(T < x)} = \frac{1}{2} = P(B_T = a),$$

and thus B_T and T are independent. Thus,

$$1 = E\left(e^{\sqrt{2\lambda}B_T}e^{-\lambda T}\right) = E\left(e^{\sqrt{2\lambda}B_T}\right)E\left(e^{-\lambda T}\right) = \frac{e^{a\sqrt{2\lambda}} + e^{-a\sqrt{2\lambda}}}{2}E\left(e^{-\lambda T}\right),$$

which gives the result.

8.39 R:

```
> trials <- 10000
> sim <- numeric(trials)
> mu = 1.5
> sig = 2
> for (i in 1:trials) {
+ idx <- seq(0,3,1/100)
+ xx <- mu*idx + sig*bm(3,300)
+ sim[i] <- tail(xx,1)
+ }
> mean(sim>4)
[1] 0.5576
```

Exact value is

$$P(X_3 > 4) = P(4.5 + 2B_3 > 4) = P(B_3 > -0.25) = 0.55738\ldots$$

8.40 R:

```
> trials <- 10000
> sim <- numeric(trials)
> for (i in 1:trials) {
+  t <- seq(0,1,length=1000)
+  b <- c(0,cumsum(rnorm(999,0,1)))/sqrt(1000)
+  bb <- b - t*b[1000]
+  sim[i] <- bb[750]  # X_{3/4}
+ }
> mean(sim <= 1/3)  # P(X_{3/4} <= 1/3)
[1] 0.7818
```

For exact value, $P(X_{3/4} \leq 1/3) = P(B_{3/4} - (3/4)B_1 \leq 1/3)$. The random variable $B_{3/4} - 3/4B_1$ has a normal distribution with mean 0 and variance

$$Var(B_{3/4} - 3/4B_1) = Var(B_{3/4}) + \frac{9}{16}Var(B_1) - 2\text{Cov}(B_{3/4}, (3/4)B_1) = \frac{3}{4} + \frac{9}{16} - \frac{9}{8} = \frac{3}{16}.$$

The desired probability is

```
> pnorm(1/3,0,sqrt(3/16))
[1] 0.7792908
```

8.41 R:

```
> trials <- 10000
> mu <- -0.85
> var <- 2.4
> sim <- numeric(trials)
> for (i in 1:trials) {
+ b <- tail(bm(2,1000), 1)
+ gbm <- 50*exp(mu*2 + sqrt(var)*b)
+ if (gbm <= 40) sim[i] <- 1
+ }
> mean(sim)
[1] 0.7512
```

Exact value is

$$P(G_2 \leq 40) = P\left(50e^{-0.85(2)+\sqrt{2.4}B_2} \leq 40\right) = P\left(B_2 \leq \frac{1.7 + \ln 0.8}{\sqrt{2.4}}\right) = 0.749\ldots$$

8.42 R: Simulate the mean and variance of the maximum of standard Brownian motion on $[0, 1]$. See Exercise 8.15 for exact values.

```
> sim <- replicate(10000,max(bm(1,1000)))
> mean(sim)
[1] 0.782218
> var(sim)
[1] 0.3685435
```

8.43 R:

```
option <- function(initial,strike,expire,interest,volat) {
alpha <- (log(strike/initial)-(interest-volat/2)*expire)/sqrt(volat)
p1 <- initial*(1-pnorm( (alpha-sqrt(volat)*expire)/sqrt(volat) ) )
p2 = exp(-interest*expire)*strike*(1-pnorm(alpha/sqrt(expire)))
return(p1-p2)
}
```

a)

```
> option(400,420,30/365,0.03,0.40)
[1] 35.3263
```

b) Price is increasing in each of the parameters, except strike price, which is decreasing.

c) Approximate volatility is 21%.

Chapter 9

9.1 The distribution is normal, with

$$E\left(\int_0^t sB_s\, ds\right) = \int_0^t sE(B_s)\, ds = 0,$$

and

$$\begin{aligned}
Var\left(\int_0^t sB_s\, ds\right) &= E\left(\left(\int_0^t sB_s\, ds\right)^2\right) = \int_{x=0}^t \int_{y=0}^t E(xB_x yB_y)\, dy\, dx \\
&= \int_{x=0}^t \int_{y=0}^x xyE(B_x B_y)\, dy\, dx + \int_{x=0}^t \int_{y=x}^t xyE(B_x B_y)\, dy\, dx \\
&= \int_{x=0}^t \int_{y=0}^x xy^2\, dy\, dx + \int_{x=0}^t \int_{y=x}^t x^2 y\, dy\, dx \\
&= \int_{x=0}^t \frac{x^4}{3}\, dx + \int_{x=0}^t x^2\left(\frac{t^2}{2} - \frac{x^2}{2}\right)\, dx \\
&= \frac{t^5}{15} + \frac{t^5}{6} - \frac{t^5}{10} = \frac{2t^5}{15}.
\end{aligned}$$

9.2 Write $X_t = \mu t + \sigma B_t$. The drift coefficient is $a(t,x) = \mu$. The diffusion coefficient is $b(t,x) = \sigma$.

9.3 Find $E(B_t^4)$ by using Ito's Lemma to evaluate $d(B_t^4)$.

By Ito's Lemma, with $g(t,x) = x^4$,

$$d(B_t^4) = 6B_t^2 dt + 4B_t^3\, dB_t,$$

which gives

$$B_t^4 = 6\int_0^t B_s^2\, ds + 4\int_0^t B_s^3\, dB_s,$$

and

$$E(B_t^4) = 6\int_0^t E(B_s^2)\, ds = 6\int_0^t s\, ds = 3t^2.$$

9.4 Use Ito's Lemma to show that

$$E(B_t^k) = \frac{k(k-1)}{2}\int_0^t E(B_s^{k-2})\, ds, \text{ for } k \geq 2.$$

Use this result to find $E(B_t^k)$, for $k = 1, \ldots, 8$.

Let $g(x) = x^k$. By Ito's Lemma

$$d(B_t^k) = \frac{k(k-1)}{2}B_t^{k-2} dt + kB_t^{k-1}\, dB_t,$$

which gives

$$B_t^k = \frac{k(k-1}{2}\int_0^t B_s^{k-2}\,ds + k\int_0^t B_s^{k-1}\,dB_s,$$

and

$$\begin{aligned} E(B_t^k) &= \frac{k(k-1)}{2}\int_0^t E(B_s^{k-2})\,ds + kE\left(\int_0^t B_s^{k-1}\,dB_s\right) \\ &= \frac{k(k-1)}{2}\int_0^t E(B_s^{k-2})\,ds. \end{aligned}$$

This gives $E(B_t^k) = 0$, for k odd, and $E(B_t^2) = t$, $E(B_t^4) = 3t^2$, $E(B_t^6) = 15t^3$, and $E(B_t^8) = 105t^4$.

9.5 In Exercise 9.3, it is shown that

$$B_t^4 = 6\int_0^t B_s^2\,ds + 4\int_0^t B_s^3\,dBs.$$

Consider

$$\begin{aligned} 4\int_0^t \left(B_s^3 - 3sB_s\right)\,dBs &= 4\int_0^t B_s^3\,dBs - 6\int_0^t 2sB_s\,dBs \\ &= \left(B_t^4 - 6\int_0^t B_s^2\,ds\right) - 6\left(tB_t^2 - \int_0^t (B_s^2 + s)\,ds\right) \\ &= B_t^4 - 6tB_t^2 + 6\int_0^t s\,ds = B_t^4 - 6tB_t^2 + 3t^2. \end{aligned}$$

This shows that $B_t^4 - 6tB_t^2 + 3t^2$ is a martingale. The second equality integration by parts is derived by Ito's Lemma in Example 9.7

9.6 Consider the SDE

$$dX_t = (1 - 2X_t)dt + 3\,dB_t.$$

a)

$$d(e^{rt}X_t) = \left(re^{rt}X_t + e^{rt}(1 - 2X_t)\right)dt + e^{rt}3\,dB_t.$$

b) Letting $r = 2$ gives

$$d(e^{2t}X_t) = e^{2t}dt + 3e^{2t}\,dB_t,$$

which gives

$$e^{2t}X_t - X_0 = \int_0^t e^{2s}\,ds + 3\int_0^t e^{2s}\,dB_s = \frac{1}{2}(e^{2t} - 1) + 3\int_0^t e^{2s}\,dB_s.$$

Thus,

$$X_t = e^{-2t}X_0 + \frac{1}{2}(1 - e^{-2t}) + 3e^{-2t}\int_0^t e^{2s}\,dB_s,$$

with

$$E(X_t) = e^{-2t}E(X_0) + \frac{1}{2}(1 - e^{-2t}) \to \frac{1}{2},\ \text{as } t \to \infty.$$

9.7 Let $g(t,x) = (x+x_0)^2$, with partial derivatives

$$\frac{\partial g}{\partial t} = 0, \quad \frac{\partial g}{\partial x} = 2(x+x_0) \quad \text{and} \quad \frac{\partial^2 g}{\partial x^2} = 2.$$

By Ito's Lemma,

$$d\left((B_t + x_0)^2\right) = dt + 2(x_0 + B_t)\, dB_t.$$

That is,

$$dX_t = dt + 2\sqrt{X_t}\, dB_t.$$

9.8 `R`: a)

```
mu <- 1
sigma <- 1/2
t <- 2
n <- 1000
xpath <- numeric(n+1)
xpath[1] <- 8  # initial value
 for (i in 1:n) {
  xpath[i+1] <- xpath[i] + (mu + sigma^2 /2)*xpath[i]*t/n
     + sigma*xpath[i]*sqrt(t/n)*rnorm(1)}
plot(seq(0,t,t/n),xpath,type="l")
```

b)

```
> t <- 2
> trials <- 5000
> n <- 1000
> mu <- 1
> sigma <- 1/2
> simlist <- numeric(trials)
> for (k in 1:trials){
+   x <- 8
+   for (i in 1:n){
+   x <- x + (mu + sigma^2 /2)*x*t/n+sigma*x*sqrt(t/n)*rnorm(1)}
+   simlist[k] <- x}
> mean(simlist)
[1] 75.88187
> var(simlist)
[1] 3629.579
```

Exact mean and variance (see Exercise 8.20) are

$$E(G_t) = G_0 e^{t(\mu+\sigma^2/2)} = 8e^{2.25} = 75.902,$$

$$Var(G_t) = G_0^2 e^{2t(\mu+\sigma^2/2)}(e^{t\sigma^2} - 1) = 64e^{4.5}(0.6487) = 3737.346.$$

9.9 `R` a)

```
> trials <- 10000
> sim <- numeric(trials)
> for (k in 1:trials) {
+ n <- 1000
+ t <- 3
+ x <- 1  # initial value
+ for (i in 1:n) {
+   x <- x + 3/n + 2*sqrt(abs(x))*sqrt(t/n)*rnorm(1)
+ }
+ sim[k] <- x
+ }
> mean(sim)  # E(X_3)
[1] 4.011273
> var(sim)      # V(X_3)
[1] 30.29936
> mean(sim < 5)  #P(X_3 < 5)
[1] 0.7293
```

Note that for the Euler-Maruyama method here, the absolute value of x is taken in the square root function. This insures that the argument is not-negative, and results in a solution which is equivalent to the original SDE.

b) Exact results are

$$E(X_3) = E((B_3 + 1)^2) = E(B_3^2) + E(2B_3) + 1 = 3 + 1 = 4,$$

$$\begin{aligned} Var(X_3) &= Var((B_3 + 1)^2) = Var(B_3^2 + 2B_3 + 1) = Var(B_3^2) + 4Var(B_3) + 2\text{Cov}(B_3^2, 2B_3) \\ &= E(B_3^4) - E(B_3^2)^2 + 4Var(B_3) + 6E(B_3^3) = 3(3)^2 - 3^2 + 4(3) = 30, \end{aligned}$$

$$P(X_3 < 5) = P(-\sqrt{5} < B_3 + 1 < \sqrt{5}) = P(-3.236 < B_3 < 1.236) = 0.7314.$$

9.10 R:

```
> trials <- 10000
> t  <- 10
> n <- 1000
> sim <- numeric(trials)
> for (k in 1:trials) {
+  x <- 1/2
+  ct <- 0
+  while (0 < x & x < 1) {
+  x <- x + sqrt(abs(x*(1-x)))*sqrt(t/n)*rnorm(1)
+  ct <- ct + 1 }
+  sim[k] <- ct * t/n
+  }
> mean(sim)
[1] 1.406156
> sd(sim)
[1] 1.059417
```

9.11
```
> trials <- 1000
> sim <- numeric(trials)
> x0 <- 0
> mu <- 1.25
> r <- 2
> sigma <- 0.2
> t <- 100
> for (k in 1:trials) {
+ n <- 1000
+ x <- x0  # initial value
+ for (i in 1:n) {
+   x <- x - r*(x-mu)*t/n + sigma*sqrt(abs(x))*sqrt(t/n)*rnorm(1)
+ }
+ sim[k] <- x
+ }
> mean(sim)
[1] 1.251721
> sd(sim)
[1] 0.1167753
```

Appendix A Getting Started with R

1.1
```
> factorial(52)
[1] 8.065818e+67
```

1.2
```
> sqrt(5^2 + 5^2)
[1] 7.071068
```

1.3
```
> (exp(1)-exp(-1))/2
[1] 1.175201
> sinh(1)
[1] 1.175201
```

3.1
```
> choose(52,5)
[1] 2598960
```

3.2
```
> choose(52,13)
[1] 635013559600
```

4.1
```
> (1:10)^2
 [1]   1   4   9  16  25  36  49  64  81 100
```

4.2
```
> 2^(1:20)
 [1]       2       4       8      16      32      64
 [7]     128     256     512    1024    2048    4096
[13]    8192   16384   32768   65536  131072  262144
[19]  524288 1048576
```

4.3
```
> choose(6,0:6)
[1]  1  6 15 20 15  6  1
```

4.4
```
> cos(2*pi*(1:5)/5)
[1]  0.309017 -0.809017 -0.809017  0.309017  1.000000
> cos(2*pi*(1:13)/13)
 [1]  0.8854560  0.5680647  0.1205367 -0.3546049
 [5] -0.7485107 -0.9709418 -0.9709418 -0.7485107
 [9] -0.3546049  0.1205367  0.5680647  0.8854560
[13]  1.0000000
```

4.5
```
> sum((1:100)^3)
[1] 25502500
```

4.6
```
> length(which((1:100)^3>10000))
[1] 79
```

5.1
```
> sample(c(0,2,5,9),5,replace=T,prob=c(0.1,0.2,0.3,0.4))
[1] 2 2 9 9 5
```

5.2
```
> x <-sample(c(0,2,5,9),10^6,replace=T,prob=c(0.1,0.2,0.3,0.4))
> mean(x==9)
[1] 0.400525
```

5.3
```
> hand <- sample(52,5)
> length(which(sample(52,5)<=4))
```

5.4
```
> x <- sample(1:20,100000,replace=T)
> mean(x>=3 & x<=7)
[1] 0.25075
```

8.1
```
> radius <- 1:100
> volume <- (4/3)*pi*radius^3
> plot(radius,volume,main="Volume as function of radius",type="l")
```

8.2
```
n <- 1000 # number of coin flips
coinflips <- sample(0:1,n,replace=TRUE)
heads <- cumsum(coinflips)
prop <- heads/(1:n)  # cumulative proportion of heads
plot(1:n,prop,type="l",xlab="Number of coins",
  ylab="Running average",
  main = "Running proportion of heads in 1000 coin flips")
abline(h = 0.5)
```

8.3
```
> n <- 2
> x <- 1:n
> sqrt((1/(n-1))*sum((x-mean(x))^2))
[1] 0.7071068
> n <- 10
> x <- 1:n
> sqrt((1/(n-1))*sum((x-mean(x))^2))
[1] 3.02765
> n <- 10000
> x <- 1:n
> sqrt((1/(n-1))*sum((x-mean(x))^2))
[1] 2886.896
```

9.1
```
> hypot <- function(a,b) sqrt(a^2+b^2)
> hypot(3,4)
[1] 5
```

9.2
```
> tetra <- function(n) mean(sample(1:4,n,replace=T))
> tetra(1)
[1] 1
> tetra(1000)
[1] 2.518
> tetra(1000000)
[1] 2.49906
```

11.1
```
> A <- matrix(c(1,3,-1,2,1,0,4,-2,3),nrow=3,byrow=T)
> A%*%A%*%A
     [,1] [,2] [,3]
[1,]    3   25  -15
[2,]   10   23  -10
```

```
[3,]   40   30    3
> t(A)
     [,1] [,2] [,3]
[1,]    1    2    4
[2,]    3    1   -2
[3,]   -1    0    3
> solve(A)
           [,1] [,2]       [,3]
[1,] -0.4285714    1 -0.1428571
[2,]  0.8571429   -1  0.2857143
[3,]  1.1428571   -2  0.7142857
> A%*%A%*%A%*%A%*%A -3*A%*%A%*%A + 7*diag(3)
     [,1] [,2] [,3]
[1,]  -73   64  -92
[2,]  -28  139 -106
[3,]  156  452 -207
```

11.2
```
mat <- matrix(runif(16,0,1),nrow=4)
eigen(mat)$values
```

11.3
```
> mat <- matrix(c(2,3,-1,1,1,1,1,2,-3),nrow=3,byrow=3)
> solve(mat,c(1,2,3))
[1]  21 -15  -4
```